The multi-dimensional Hamiltonian Structures in the Whitham method.

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Abstract

In this paper we consider the averaging of local field-theoretic Poisson brackets in the multi-dimensional case. As a result, we construct a local Poisson bracket for the regular Whitham system in the multidimensional situation. The procedure is based on the procedure of averaging of local conservation laws and follows the Dubrovin - Novikov scheme of the bracket averaging suggested in one-dimensional case. However, the features of the phase space of modulated parameters in higher dimensions lead to a different natural class of the averaged brackets in comparison with the one-dimensional situation. Here we suggest a direct procedure of construction of the bracket for the Whitham system for $d > 1$ and discuss the conditions of applicability of the corresponding scheme. At the end, we discuss canonical forms of the averaged Poisson bracket in the multidimensional case.

1 Introduction.

We consider here the Whitham averaging method ([35, 36, 37]) associated with slow modulations of periodic or quasiperiodic m-phase solutions of nonlinear systems

$$F^{i}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{t}, \boldsymbol{\varphi}_{x^1}, \dots, \boldsymbol{\varphi}_{x^d}, \dots) \; = \; 0 \quad , \quad i = 1, \dots, n \;\; , \;\; \boldsymbol{\varphi} = (\varphi^1, \dots, \varphi^n) \tag{1.1}$$

which are usually represented in the form

$$\varphi^{i}(\mathbf{x}, t) \; = \; \Phi^{i} \left(\mathbf{k}_{1}(\mathbf{U})\, x^{1} \, + \, \dots \, + \, \mathbf{k}_{d}(\mathbf{U})\, x^{d} \, + \, \boldsymbol{\omega}(\mathbf{U})\, t \, + \, \boldsymbol{\theta}_{0}, \, \mathbf{U} \right) \tag{1.2}$$

We are going to consider here systems with d spatial variables $(x^1, \dots, x^d)$ and one time variable t. In these notations the functions $\mathbf{k}_q(\mathbf{U})$ and $\boldsymbol{\omega}(\mathbf{U})$ play the role of the "wave numbers" and "frequencies" of m-phase solutions, while the parameters $\boldsymbol{\theta}_0$ represent the "initial phase shifts". The parameters $\mathbf{U} = (U^1, \dots, U^N)$ can be chosen in an arbitrary way, we just assume that they do not change under shifts of the initial phases of solutions $\boldsymbol{\theta}_0$.

The functions $\Phi^i(\boldsymbol{\theta})$ satisfy the system

$$F^{i} \left(\boldsymbol{\Phi}, \omega^{\alpha} \boldsymbol{\Phi}_{\theta^{\alpha}}, k_{1}^{\beta_1} \boldsymbol{\Phi}_{\theta^{\beta_1}}, \dots, k_{d}^{\beta_d} \boldsymbol{\Phi}_{\theta^{\beta_d}}, \dots \right) \; \equiv \; 0 \quad , \quad i = 1, \dots, n \tag{1.3}$$

and we have to choose for each value of $\mathbf{U}$ some function $\mathbf{\Phi}(\boldsymbol{\theta},\mathbf{U})$ as having "zero initial phase shift". The corresponding set of m-phase solutions of (1.1) can be then represented in the form (1.2). For m-phase solutions of (1.1) we have in this case $\mathbf{k}_p(\mathbf{U}) = (k_p^1(\mathbf{U}),\dots,k_p^m(\mathbf{U}))$, $\boldsymbol{\omega}(\mathbf{U}) = (\omega^1(\mathbf{U}),\dots,\omega^m(\mathbf{U}))$, $\boldsymbol{\theta}_0 = (\theta_0^1,\dots,\theta_0^m)$, where $\mathbf{U} = (U^1,\dots,U^N)$ are the parameters of a solution. We will require also that all the functions $\Phi^i(\boldsymbol{\theta},\mathbf{U})$ are 2π-periodic with respect to each θ^α, $\alpha = 1,\dots,m$.

Consider a set Λ of functions $\mathbf{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\mathbf{U})$, depending smoothly on the parameters $\mathbf{U}$ and satisfying system (1.3) for all $\mathbf{U}$.

In the Whitham approach the parameters $\mathbf{U}$ and $\boldsymbol{\theta}_0$ become slowly varying functions of $\mathbf{x}$ and t: $\mathbf{U} = \mathbf{U}(\mathbf{X},T)$, $\boldsymbol{\theta}_0 = \boldsymbol{\theta}_0(\mathbf{X},T)$, where $\mathbf{x} = (x^1,\dots,x^d)$, $\mathbf{X} = (X^1,\dots,X^d)$, $X^q = \epsilon x^q$, $T = \epsilon t$ ($\epsilon \to 0$).

For the construction of the corresponding asymptotic solution the functions $\mathbf{U}(\mathbf{X},T)$ must satisfy some system of differential equations (the Whitham system). In the simplest case (see [20]), we try to find asymptotic solutions

$$\varphi^i(\boldsymbol{\theta},\mathbf{X},T,\epsilon) \;=\; \sum_{k\geq 0} \Psi^i_{(k)}\left(\frac{\mathbf{S}(\mathbf{X},T)}{\epsilon} + \boldsymbol{\theta},\, \mathbf{X},\, T\right)\,\epsilon^k \tag{1.4}$$

with 2π-periodic in $\boldsymbol{\theta}$ functions $\mathbf{\Psi}_{(k)}$ satisfying system (1.1), i.e.

$$F^i\left(\boldsymbol{\varphi}, \epsilon\,\boldsymbol{\varphi}_T, \epsilon\,\boldsymbol{\varphi}_{X^1},\dots,\epsilon\,\boldsymbol{\varphi}_{X^d},\dots\right) \;=\; 0 \quad , \quad i = 1,\dots,n$$

The function $\mathbf{S}(\mathbf{X},T) = (S^1(\mathbf{X},T),\dots,S^m(\mathbf{X},T))$ is called the "modulated phase" of solution (1.4).

Assume now that the function $\mathbf{\Psi}_{(0)}(\boldsymbol{\theta},\mathbf{X},T)$ belongs to the family Λ of m-phase solutions of (1.1) for all $\mathbf{X}$ and T. We have then

$$\mathbf{\Psi}_{(0)}(\boldsymbol{\theta},\mathbf{X},T) \;=\; \mathbf{\Phi}\left(\boldsymbol{\theta} + \boldsymbol{\theta}_0(\mathbf{X},T), \mathbf{U}(\mathbf{X},T)\right) \tag{1.5}$$

and

$$S^\alpha_T(\mathbf{X},T) \;=\; \omega^\alpha(\mathbf{U}(\mathbf{X},T)) \;\;,\;\;\; S^\alpha_{X^p}(\mathbf{X},T) \;=\; k^\alpha_p(\mathbf{U}(\mathbf{X},T))$$

as follows after the substitution of (1.4) into system (1.1).

In the simplest case the functions $\mathbf{\Psi}_{(k)}(\boldsymbol{\theta},\mathbf{X},T)$ are determined from the linear systems

$$\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(\mathbf{X},T)\;\Psi^j_{(k)}(\boldsymbol{\theta},\mathbf{X},T) \;=\; f^i_{(k)}(\boldsymbol{\theta},\mathbf{X},T) \tag{1.6}$$

where $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(\mathbf{X},T)$ is a linear operator defined by the linearization of system (1.3) on the solution (1.5). The solubility conditions of systems (1.6) in the space of periodic functions can be written as the conditions of orthogonality of the functions $\mathbf{f}_{(k)}(\boldsymbol{\theta},\mathbf{X},T)$ to all the "left eigenvectors" (the eigenvectors of the adjoint operator) of the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(\mathbf{X},T)$ corresponding to zero eigenvalue.

We should say, however, that the solubility conditions of systems (1.6) can actually be quite complicated in general multi-phase case, since the eigenspaces of the operators $\hat{L}_{[\mathbf{U},\boldsymbol{\theta}_0]}$ and $\hat{L}^\dagger_{[\mathbf{U},\boldsymbol{\theta}_0]}$ on the space of 2π-periodic functions can be rather nontrivial in the multi-phase situation. Thus, even the dimension of the kernels of $\hat{L}_{[\mathbf{U},\boldsymbol{\theta}_0]}$ and $\hat{L}^\dagger_{[\mathbf{U},\boldsymbol{\theta}_0]}$ can depend in a highly nontrivial way on the values of $\mathbf{U}$. In general, the picture arising in the $\mathbf{U}$-space can be rather complicated. As a result, the determination of the next corrections from systems (1.6) is impossible in general multiphase situation

and the corrections to the main approximation (1.5) have more complicated and rather nontrivial form ([6, 7, 8]).

These difficulties do not arise commonly in the single-phase situation ($m = 1$) where the behavior of eigenvectors of $\hat{L}_{[\mathbf{U},\boldsymbol{\theta}_0]}$ and $\hat{L}^{\dagger}_{[\mathbf{U},\boldsymbol{\theta}_0]}$, as a rule, is quite regular. The solubility conditions of system (1.6) for $k = 1$

$$\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(\mathbf{X},T)\ \Psi^j_{(1)}(\boldsymbol{\theta},\mathbf{X},T) \ = \ f^i_{(1)}(\boldsymbol{\theta},\mathbf{X},T) \tag{1.7}$$

with the relations

$$k_{pT} = \omega_{X^p} \quad , \quad k_{pX^l} = k_{lX^p} \quad , \quad p,l = 1,\ldots,d$$

define in this case the Whitham system for the single-phase solutions of (1.1) which plays the central role in considering the slow modulations.

For the multi-phase solutions the Whitham system is usually defined by the orthogonality conditions of the right-hand part of (1.7) to the maximal set of "regular" left eigenvectors corresponding to zero eigenvalues which are defined for all values of $\mathbf{U}$ and depend smoothly on $\mathbf{U}$.

As mentioned above, the construction of asymptotic series (1.4) in the multi-phase case is impossible in general situation (see [6, 7, 8]). Nevertheless, the Whitham system (1.13) and the leading term of the expansion (1.4) play the major role in consideration of modulated solutions also in this case, representing the main approximation for the corresponding modulated solutions. The corrections to the main term have in general more nontrivial form than (1.4), but they also tend to zero in the limit $\epsilon \to 0$ ([6, 7, 8]).

Let us give here just some incomplete list of the classical papers devoted to the foundations of the Whitham method [1, 4, 5, 6, 7, 8, 9, 11, 12, 17, 18, 19, 20, 29, 30, 31, 35, 36, 37]. We will be interested here only in Hamiltonian aspects of the multi-dimensional Whitham method. In general, the article represents the technique and methods developed in the paper [25], applied to the multi-dimensional case. However, as we will see, the features of the phase space of modulated parameters in higher dimensions lead to a different natural class of the averaged brackets in comparison with the one-dimensional situation.

In the remaining part of the Introduction we will give the definition of "regular" Whitham system for complete regular family of m-phase solutions which will be used everywhere below.

Let us use for simplicity the notation Λ both for the family of the functions $\mathbf{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\mathbf{U})$ and the corresponding family of m-phase solutions of system (1.1), such that we will denote by Λ both the parameter-dependent families of the 2π-periodic in all θ^α functions $\mathbf{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\mathbf{U})$ and $\boldsymbol{\varphi}_{[\mathbf{U},\boldsymbol{\theta}_0]}(\mathbf{x}) = \mathbf{\Phi}(\mathbf{k}_q(\mathbf{U})\, x^q + \boldsymbol{\theta}_0, \mathbf{U})$. We will assume everywhere below that the family Λ represents a smooth family of m-phase solutions of system (1.1) in the sense discussed above.

It is generally assumed that the parameters k^α_p, ω^α are independent on the family Λ, such that the full family of the m-phase solutions of (1.1) depends on $N = (d+1)m + s$, ($s \geq 0$) parameters U^ν and m initials phase shifts θ^α_0. In this case it is convenient to represent the parameters $\mathbf{U}$ in the form $\mathbf{U} = (\mathbf{k}_1,\ldots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n})$, where $\mathbf{k}_p$ represents the wave numbers, $\boldsymbol{\omega}$ - the frequencies of the m-phase solutions and $\mathbf{n} = (n^1,\ldots,n^s)$ - some additional parameters (if any).

It is easy to see that the functions $\mathbf{\Phi}_{\theta^\alpha}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\mathbf{k}_1,\ldots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n})$, $\alpha = 1,\ldots,m$, $\mathbf{\Phi}_{n^l}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\mathbf{k}_1,\ldots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n})$, $l = 1,\ldots,s$, belong to the kernel of the operator $\hat{L}^i_{j[\mathbf{k}_1,\ldots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n},\boldsymbol{\theta}_0]}$. In the regular case it is natural to assume that the set of the functions $(\mathbf{\Phi}_{\theta^\alpha},\ \mathbf{\Phi}_{n^l})$ represents the maximal

linearly independent set of the kernel vectors of the operator $\hat{L}$ regularly depending on the parameters $(\mathbf{k}_1, \ldots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$. For the construction of the "regular" Whitham system we have to require the following property of regularity and completeness of the family of m-phase solutions of system (1.1):

Definition 1.1.

We call a family Λ a complete regular family of m-phase solutions of system (1.1) if:

1) The values $\mathbf{k}_p = (k_p^1, \ldots, k_p^m)$, $\boldsymbol{\omega} = (\omega^1, \ldots, \omega^m)$ represent independent parameters on the family Λ, such that the total set of parameters of the m-phase solutions can be represented in the form $(\mathbf{U}, \boldsymbol{\theta}_0) = (\mathbf{k}_1, \ldots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}, \boldsymbol{\theta}_0)$;

2) The functions $\mathbf{\Phi}_{\theta^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \ldots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$, $\mathbf{\Phi}_{n^l}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \ldots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ are linearly independent and give the maximal linearly independent set among the kernel vectors of the operator $\hat{L}^i_{j[\mathbf{k}_1,\ldots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n},\boldsymbol{\theta}_0]}$, smoothly depending on the parameters $(\mathbf{k}_1, \ldots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ on the whole set of parameters;

3) The operator $\hat{L}^i_{j[\mathbf{k}_1,\ldots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n},\boldsymbol{\theta}_0]}$ has exactly $m+s$ linearly independent left eigenvectors with zero eigenvalue

$$\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) \;=\; \boldsymbol{\kappa}^{(q)}_{[\mathbf{k}_1,\ldots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n}]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) \quad , \quad q = 1, \ldots, m+s$$

among the vectors smoothly depending on the parameters $(\mathbf{k}_1, \ldots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ on the whole set of parameters.

By definition, we will call the regular Whitham system for a complete regular family of m-phase solutions of (1.1) the conditions of orthogonality of the discrepancy $\mathbf{f}_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T)$ to the functions $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}(\mathbf{X},T)]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(\mathbf{X}, T))$

$$\int_0^{2\pi} \!\!\ldots \int_0^{2\pi} \kappa^{(q)}_{[\mathbf{U}(\mathbf{X},T)]\,i}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(\mathbf{X}, T)) \; f^i_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T) \; \frac{d^m\theta}{(2\pi)^m} \;=\; 0 \quad , \quad q = 1, \ldots, m+s \qquad (1.8)$$

with the compatibility conditions

$$k^\alpha_{pT} \,=\, \omega^\alpha_{X^p} \quad , \quad k^\alpha_{pX^l} \,=\, k^\alpha_{lX^p} \quad , \quad \alpha = 1, \ldots, m \;\; , \quad p, l = 1, \ldots, d \qquad (1.9)$$

System (1.8) - (1.9) gives $md(d+1)/2 + (m+s)$ conditions at every $\mathbf{X}$ and T for the parameters of the zero approximation $\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, \mathbf{X}, T)$.

It is well known that the Whitham system does not include the parameters $\theta^\alpha_0(\mathbf{X}, T)$ and provides restrictions only to the parameters $U^\nu(\mathbf{X}, T)$ of the zero approximation. Let us prove here a simple lemma which confirms this property under the conditions formulated above.

Lemma 1.1.

Under the regularity conditions formulated above the orthogonality conditions (1.8) do not contain the functions $\theta^\alpha_0(\mathbf{X}, T)$ and give constraints only to the functions $U^\nu(\mathbf{X}, T)$, having the form

$$C^{(q)}_\nu(\mathbf{U})\, U^\nu_T \;-\; D^{(q)p}_\nu(\mathbf{U})\, U^\nu_{X^p} \;=\; 0 \quad , \quad q = 1, \ldots, m+s \qquad (1.10)$$

(with some functions $C^{(q)}_\nu(\mathbf{U})$, $D^{(q)p}_\nu(\mathbf{U})$, $\nu = 1, \ldots, N$, $p = 1, \ldots, d$).

Proof.

Let us write down the part $\mathbf{f}'_{(1)}$ of the function $\mathbf{f}_{(1)}$, which contains the derivatives $\theta^{\beta}_{0T}(\mathbf{X},T)$ and $\theta^{\beta}_{0X^p}(\mathbf{X},T)$. We have

$$f'^{i}_{(1)}(\boldsymbol{\theta},\mathbf{X},T) \; = \; - \, {\partial F^{i} \over \partial \varphi^{j}_{t}} \left(\boldsymbol{\Psi}_{(0)}, \, \dots \right) \, \Psi^{j}_{(0)\theta^{\beta}} \, \theta^{\beta}_{0T} \; - \; {\partial F^{i} \over \partial \varphi^{j}_{x^p}} \left(\boldsymbol{\Psi}_{(0)}, \, \dots \right) \, \Psi^{j}_{(0)\theta^{\beta}} \, \theta^{\beta}_{0X^p} \; -$$

$$- \, {\partial F^{i} \over \partial \varphi^{j}_{tt}} \left(\boldsymbol{\Psi}_{(0)}, \, \dots \right) \, 2\, \omega^{\alpha}(\mathbf{X},T) \, \Psi^{j}_{(0)\theta^{\alpha}\theta^{\beta}} \, \theta^{\beta}_{0T} \; - \; {\partial F^{i} \over \partial \varphi^{j}_{x^p x^l}} \left(\boldsymbol{\Psi}_{(0)}, \, \dots \right) \, 2\, k^{\alpha}_{p}(\mathbf{X},T) \, \Psi^{j}_{(0)\theta^{\alpha}\theta^{\beta}} \, \theta^{\beta}_{0X^l} \; - \; \dots$$

Let us choose the parameters $\mathbf{U}$ in the form

$$\mathbf{U} = (k^{1}_{p}, \dots, k^{m}_{p}, \omega^{1}, \dots, \omega^{m}, n^{1}, \dots, n^{s})$$

We can write then

$$f'^{i}_{(1)}(\boldsymbol{\theta},\mathbf{X},T) \; = \; \left[- {\partial \over \partial \omega^{\beta}} \, F^{i} \left(\boldsymbol{\Phi}(\boldsymbol{\theta} + \boldsymbol{\theta}_{0}, \mathbf{U}), \dots \right) \; + \; \hat{L}^{i}_{j} \, {\partial \over \partial \omega^{\beta}} \, \Phi^{j}(\boldsymbol{\theta} + \boldsymbol{\theta}_{0}, \mathbf{U}) \right] \, \theta^{\beta}_{0T} \, +$$

$$+ \, \left[- {\partial \over \partial k^{\beta}_{p}} \, F^{i} \left(\boldsymbol{\Phi}(\boldsymbol{\theta} + \boldsymbol{\theta}_{0}, \mathbf{U}), \dots \right) \; + \; \hat{L}^{i}_{j} \, {\partial \over \partial k^{\beta}_{p}} \, \Phi^{j}(\boldsymbol{\theta} + \boldsymbol{\theta}_{0}, \mathbf{U}) \right] \, \theta^{\beta}_{0X^p}$$

The total derivatives $\partial F^{i} / \partial \omega^{\beta}$ and $\partial F^{i} / \partial k^{\beta}_{p}$ are identically equal to zero on Λ according to (1.3). We have then

$$\int_{0}^{2\pi} \!\! \dots \int_{0}^{2\pi} \kappa^{(q)}_{[\mathbf{U}(\mathbf{X},T)]\, i}(\boldsymbol{\theta} \; + \; \boldsymbol{\theta}_{0}(\mathbf{X},T)) \, f'^{i}_{(1)}(\boldsymbol{\theta},\mathbf{X},T) \, {d^{m}\theta \over (2\pi)^{m}} \;\; \equiv \;\; 0$$

since $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}(\mathbf{X},T)]}(\boldsymbol{\theta} + \boldsymbol{\theta}_{0}(\mathbf{X},T))$ are left eigenvectors of $\hat{L}$ with the zero eigenvalue.

It is not difficult to see also that all the $\boldsymbol{\theta}_{0}(\mathbf{X},T)$ in the arguments of $\boldsymbol{\Phi}$ and $\boldsymbol{\kappa}^{(q)}$ disappear after the integration with respect to $\boldsymbol{\theta}$, so we get the statement of the Lemma.

Lemma 1.1 is proved.

Conditions (1.8) together with the compatibility conditions

$$k^{\alpha}_{pT} \; = \; \omega^{\alpha}_{X^p} \quad , \quad \alpha = 1, \dots, m \;\; , \;\; p = 1, \dots, d$$

give $(m+s) + md = m(d+1) + s$ restrictions on the functions $\mathbf{U}(\mathbf{X},T)$ which is exactly equal to the number of the parameters U^{ν}.

Let us call the system

$$\begin{gathered} C^{(q)}_{\nu}(\mathbf{U}) \, U^{\nu}_{T} \; = \; D^{(q)p}_{\nu}(\mathbf{U}) \, U^{\nu}_{X^p} \;\; , \quad q \; = \; 1, \dots, m+s \\ k^{\alpha}_{pT} \; = \; \omega^{\alpha}_{X^p} \quad , \quad \alpha = 1, \dots, m \;\; , \;\; p = 1, \dots, d \end{gathered} \tag{1.11}$$

the evolutionary part of a regular Whitham system for a complete regular family Λ, while the restrictions

$$k^{\alpha}_{pX^l} \; = \; k^{\alpha}_{lX^p} \quad , \quad \alpha = 1, \dots, m \;\; , \;\; p, l = 1, \dots, d \tag{1.12}$$

will be considered as additional constraints for the evolutionary system (1.11). It is easy to see that the constraints (1.12) are compatible with the evolutionary system (1.11) in the sense that the restrictions (1.12) are conserved by system (1.11) being imposed at the initial time.

In generic case the derivatives U^{ν}_{T} can be expressed in terms of $U^{\mu}_{X^p}$ from system (1.11) and the evolutionary part of a regular Whitham system can be written in the form

$$U^{\nu}_{T} \;=\; V^{\nu p}_{\mu}(\mathbf{U})\, U^{\mu}_{X^p} \tag{1.13}$$

Thus, system (1.13) represents a homogeneous quasi-linear system of Hydrodynamic Type connecting the derivatives of the slow modulated parameters.

2 Lagrangian and Hamiltonian formulations of the Whitham method.

Together with the formulation of Whitham's method the Lagrangian structure of the equations of slow modulations was proposed ([35, 36, 37]). The method of averaging of Lagrangian function introduced by Whitham can be formulated in the following way. We assume that the original system (1.1) is lagrangian with the local action of the form

$$S \;=\; \int L\left(\boldsymbol{\varphi}, \boldsymbol{\varphi}_t, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{tt}, \boldsymbol{\varphi}_{\mathbf{x}t}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots\right)\, d^d x\, dt$$

such that the functions F^i have the form

$$F^i(\boldsymbol{\varphi}, \boldsymbol{\varphi}_t, \boldsymbol{\varphi}_{\mathbf{x}}, \dots) \;=\; \frac{\delta S}{\delta \varphi^i(\mathbf{x}, t)} \;=\; \frac{\partial L}{\partial \varphi^i} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \varphi^i_t} - \frac{\partial}{\partial x^p}\frac{\partial L}{\partial \varphi^i_{x^p}} + \dots$$

Let us assume for simplicity that the parameters $(\mathbf{k}_p, \boldsymbol{\omega}) = (k^1_p, \dots, k^m_p, \omega^1, \dots, \omega^m)$ give the complete set of independent parameters on the family of m-phase solutions (excluding the initial phase shifts), such that the number of parameters U^{ν} is equal to $m(d+1)$.

The linearized operator $\hat{L}^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\boldsymbol{\theta}_0]}(\mathbf{X}, T)$ in (1.6) is given now by the distribution

$$L^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\boldsymbol{\theta}_0]}(\boldsymbol{\theta}, \boldsymbol{\theta}') \;=\; \frac{\delta^2 S}{\delta \Phi^i(\boldsymbol{\theta}) \delta \Phi^j(\boldsymbol{\theta}')}$$

where

$$S \;=\; \int_0^{2\pi} \!\!\dots \int_0^{2\pi} L\left(\mathbf{\Phi},\, \omega^{\alpha}\, \mathbf{\Phi}_{\theta^{\alpha}},\, k^{\beta_1}_1\, \mathbf{\Phi}_{\theta^{\beta_1}}, \dots,\, k^{\beta_d}_d\, \mathbf{\Phi}_{\theta^{\beta_d}}, \dots\right)\, \frac{d^m\theta}{(2\pi)^m}$$

and is a self-adjoint operator.

Throughout the paper we will always understand the integration with respect to $\boldsymbol{\theta}$ as the averaging procedure. For this reason, all the integrals over $d^m\theta$ will be defined with the factor $1/(2\pi)^m$. In particular, we will also assume that the variation derivatives of the type $\delta S/\delta\varphi^i(\boldsymbol{\theta})$ are defined in the way that

$$\delta\, S \;\equiv\; \int_0^{2\pi} \!\!\dots \int_0^{2\pi} \frac{\delta\, S}{\delta \varphi^i(\boldsymbol{\theta})}\, \delta \varphi^i(\boldsymbol{\theta})\, \frac{d^m\theta}{(2\pi)^m}$$

on the space of 2π-periodic in $\boldsymbol{\theta}$ functions.

We also define here the delta function $\delta(\boldsymbol{\theta} - \boldsymbol{\theta}')$ and its higher derivatives $\delta_{\theta^{\alpha_1}\dots\theta^{\alpha_s}}(\boldsymbol{\theta} - \boldsymbol{\theta}')$ on the space of 2π-periodic functions by the formula

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi}\delta_{\theta^{\alpha_1}\dots\theta^{\alpha_s}}(\boldsymbol{\theta}-\boldsymbol{\theta}')\;\psi(\boldsymbol{\theta}')\;\frac{d^m\theta'}{(2\pi)^m}\;\;\equiv\;\;\psi_{\theta^{\alpha_1}\dots\theta^{\alpha_s}}(\boldsymbol{\theta})$$

The functions $\mathbf{\Phi}_{\theta^\alpha}$, $\alpha=1,\dots,m$ represent both the left and the right eigenfunctions of the operator $\hat{L}^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\boldsymbol{\theta}_0]}(\mathbf{X},T)$, corresponding to the zero eigenvalue.

Under the assumption that the family of the m-phase solutions Λ is a complete regular family of m-phase solutions of (1.1) we assume that the functions $\mathbf{\Phi}_{\theta^\alpha}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega})$ are linearly independent and give the maximal linearly independent set among the kernel vectors of the operator $\hat{L}^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\boldsymbol{\theta}_0]}$ smoothly depending on the parameters $(\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega})$. The regular Whitham system is given then by the conditions $k^\alpha_{pT}=\omega^\alpha_{X^p}$, $k^\alpha_{pX^l}=k^\alpha_{lX^p}$, and m conditions of orthogonality of the function $\mathbf{f}_{(1)}(\boldsymbol{\theta},\mathbf{X},T)$ to the functions $\mathbf{\Phi}_{\theta^\alpha}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega})$.

According to the Whitham procedure the Whitham system on the parameters $(\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega})$ is obtained from the condition of extremality of the action

$$\Sigma^{(0)}\left[\mathbf{S}\right]\;=\;\int\!\!\int_0^{2\pi}\!\!\dots\int_0^{2\pi}L\left(\mathbf{\Phi},\,S^\alpha_T\,\mathbf{\Phi}_{\theta^\alpha},\,S^{\beta_1}_{X^1}\,\mathbf{\Phi}_{\theta^{\beta_1}},\dots,\,S^{\beta_d}_{X^d}\,\mathbf{\Phi}_{\theta^{\beta_d}},\dots\right)\;\frac{d^m\theta}{(2\pi)^m}\,d^dX\,dT \tag{2.1}$$

under the conditions $k^\alpha_p=S^\alpha_{X^p}$, $\omega^\alpha=S^\alpha_T$.

The conditions $k^\alpha_{pT}=\omega^\alpha_{X^p}$, $k^\alpha_{pX^l}=k^\alpha_{lX^p}$, and $\delta\Sigma/\delta S^\alpha(\mathbf{X},T)=0$ give a system of $md(d+1)/2+m$ equations on the parameters $(\mathbf{k}_p,\boldsymbol{\omega})$.

It is not difficult to see that the system given by the variation of the "averaged" action coincides with the conditions of orthogonality of the function $\mathbf{f}_{(1)}(\boldsymbol{\theta},\mathbf{X},T)$ to the functions $\mathbf{\Phi}_{\theta^\alpha}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega})$. Indeed, let us consider the action

$$\Sigma\left[\mathbf{S},\boldsymbol{\varphi},\epsilon\right]\;=\;\int L\left(\boldsymbol{\varphi}\left(\frac{\mathbf{S}(\mathbf{X},T)}{\epsilon}+\boldsymbol{\theta},\mathbf{X},T\right),\epsilon\frac{\partial}{\partial T}\boldsymbol{\varphi}\left(\frac{\mathbf{S}(\mathbf{X},T)}{\epsilon}+\boldsymbol{\theta},\mathbf{X},T\right),\dots\right)\frac{d^m\theta}{(2\pi)^m}\,d^dXdT\;=$$

$$=\;\;\Sigma^{(0)}\left[\mathbf{S},\boldsymbol{\varphi}\right]\;+\;\epsilon\,\Sigma^{(1)}\left[\mathbf{S},\boldsymbol{\varphi}\right]\;+\;\epsilon^2\,\Sigma^{(2)}\left[\mathbf{S},\boldsymbol{\varphi}\right]\;+\;\dots$$

defined on the functions $\boldsymbol{\varphi}(\boldsymbol{\theta},\mathbf{X},T)$, 2π-periodic in each θ^α. Taking into account the relation

$$\frac{\delta\Sigma}{\delta S^\alpha(\mathbf{X},T)}\;=\;\epsilon^{-1}\int_0^{2\pi}\!\!\dots\int_0^{2\pi}\varphi^i_{\theta^\alpha}\left(\boldsymbol{\theta},\mathbf{X},T\right)\;\frac{\delta\Sigma}{\delta\varphi^i(\boldsymbol{\theta},\mathbf{X},T)}\;\frac{d^m\theta}{(2\pi)^m} \tag{2.2}$$

and the invariance of the action with respect to the shifts

$$\mathbf{S}(\mathbf{X},T)\;\;\rightarrow\;\;\mathbf{S}(\mathbf{X},T)\;+\;\Delta\,\mathbf{S}$$

it is easy to see that

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi}\varphi^i_{\theta^\alpha}\left(\boldsymbol{\theta},\mathbf{X},T\right)\;\frac{\delta\Sigma^{(0)}}{\delta\varphi^i(\boldsymbol{\theta},\mathbf{X},T)}\;\frac{d^m\theta}{(2\pi)^m}\;\;\equiv\;\;0 \tag{2.3}$$

$$\frac{\delta\Sigma^{(0)}}{\delta S^\alpha(\mathbf{X},T)}\;=\;\int_0^{2\pi}\!\!\dots\int_0^{2\pi}\varphi^i_{\theta^\alpha}\left(\boldsymbol{\theta},\mathbf{X},T\right)\;\frac{\delta\Sigma^{(1)}}{\delta\varphi^i(\boldsymbol{\theta},\mathbf{X},T)}\;\frac{d^m\theta}{(2\pi)^m} \tag{2.4}$$

etc.

Substituting the functions $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}, T)$ in the form $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}, T) = \boldsymbol{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_0, \mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{S}_T)$ in the relations above, we can see that we have to include now the additional dependence of the functions $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}, T)$ on $\mathbf{S}_{\mathbf{X}}$ and $\mathbf{S}_T$ in relation (2.4). However, due to the relation

$$\frac{\delta \Sigma^{(0)}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X}, T)} \equiv 0$$

on the family Λ, relation (2.4) will not change in this situation. Taking also into account the equality

$$f^i_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T) = \frac{\delta \Sigma^{(1)}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X}, T)}$$

we get the required statement.

Under the assumption of the completeness and regularity of the family Λ we can see then that the averaged action (2.1) defines a lagrangian structure of the regular Whitham system in general multiphase case. We should say also that the cases with additional parameters $\mathbf{n}$, as a rule, can be also included into the scheme described above with the aid of the Whitham "pseudo-phases" ([37]).

In this paper we are going to consider the Hamiltonian formulation of the Whitham equations. Let us consider then another approach to the construction of the regular Whitham system which is connected with the method of averaging of conservation laws. According to further consideration of the Hamiltonian structure of the Whitham equations we will assume now that system (1.1) is written in an evolutionary form

$$\varphi^i_t = F^i(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) \tag{2.5}$$

The families of the m-phase solutions of (2.5) are defined then by solutions of the system

$$\omega^\alpha \varphi^i_{\theta^\alpha} = F^i\left(\boldsymbol{\varphi}, k^{\beta_1}_1 \boldsymbol{\varphi}_{\theta^{\beta_1}}, \dots, k^{\beta_d}_d \boldsymbol{\varphi}_{\theta^{\beta_d}}, \dots\right) \tag{2.6}$$

on the space of 2π-periodic in each θ^α functions $\boldsymbol{\varphi}(\boldsymbol{\theta})$.

We will assume that the conservation laws of system (2.5) have the form

$$P^\nu_t(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) = Q^{\nu 1}_{x^1}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) + \dots + Q^{\nu d}_{x^d}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) \tag{2.7}$$

such that the values

$$I^\nu = \int P^\nu(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots)\, d^d x$$

represent translationally invariant conservative quantities for system (2.5) in the case of the rapidly decreasing at infinity functions $\boldsymbol{\varphi}(\mathbf{x})$. We can also define the conservation laws for system (2.5) in the periodic case with fixed periods $K_1, \dots, K_d$.

$$I^\nu = \frac{1}{K_1 \dots K_d} \int_0^{K_1} \dots \int_0^{K_d} P^\nu(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots)\, d^d x$$

or in the quasiperiodic case

$$I^\nu = \lim_{K\to\infty} \frac{1}{(2K)^d} \int_{-K}^{K} \dots \int_{-K}^{K} P^\nu(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots)\, d^d x$$

It is natural also to define the variation derivatives of the functionals I^{ν} with respect to the variations of $\boldsymbol{\varphi}(\mathbf{x})$ having the same periodic or quasiperiodic properties as the original functions. Easy to see then that the standard Euler - Lagrange expressions for the variation derivatives can be used in this case.

Let us write the functionals I^{ν} in the general form

$$I^{\nu} \;=\; \int P^{\nu}\left(\boldsymbol{\varphi},\, \boldsymbol{\varphi}_{\mathbf{x}},\, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots\right)\, d^d x \tag{2.8}$$

assuming the appropriate definition in the corresponding situations.

Let us define a quasiperiodic function $\boldsymbol{\varphi}(\mathbf{x})$ with fixed quasiperiods $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ as function $\boldsymbol{\varphi}(\mathbf{x})$ on $\mathbb{R}^d$ coming from a smooth periodic function $\boldsymbol{\varphi}(\boldsymbol{\theta})$ on the torus $\mathbb{T}^m$:

$$\boldsymbol{\varphi}(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\theta}_0) \quad \rightarrow \quad \boldsymbol{\varphi}(x^1, \dots, x^d)$$

Let us define the functionals

$$J^{\nu} \;=\; \int_0^{2\pi}\!\!\dots\int_0^{2\pi} P^{\nu}\left(\boldsymbol{\varphi},\, k_1^{\beta_1}\, \boldsymbol{\varphi}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d}\, \boldsymbol{\varphi}_{\theta^{\beta_d}}, \dots\right) \frac{d^m\theta}{(2\pi)^m} \tag{2.9}$$

on the space of 2π-periodic in $\boldsymbol{\theta}$ functions.

It's not difficult to see that the functions

$$\zeta^{(\nu)}_{i[\mathbf{U}]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) \;=\; \left.\frac{\delta J^{\nu}}{\delta \varphi^i(\boldsymbol{\theta})}\right|_{\boldsymbol{\varphi}(\boldsymbol{\theta}) = \boldsymbol{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_0, \mathbf{U})} \tag{2.10}$$

represent left eigenvectors of the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}$ with zero eigenvalues, regularly depending on parameters $\mathbf{U}$ on a fixed smooth family Λ.

Indeed, the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}$ is defined in this case by the distribution

$$L^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(\boldsymbol{\theta}, \boldsymbol{\theta}') \;=\; \delta^i_j\, \omega^{\alpha}\, \delta_{\theta^{\alpha}}(\boldsymbol{\theta} - \boldsymbol{\theta}') \;-\; \left.\frac{\delta F^i(\boldsymbol{\varphi},\, k_1^{\beta_1}\, \boldsymbol{\varphi}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d}\, \boldsymbol{\varphi}_{\theta^{\beta_d}}, \dots)}{\delta \varphi^j(\boldsymbol{\theta}')}\right|_{\boldsymbol{\varphi}(\boldsymbol{\theta}) = \boldsymbol{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_0, \mathbf{U})}$$

We have

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \frac{\delta J^{\nu}}{\delta \varphi^i(\boldsymbol{\theta})} \left(\omega^{\alpha}\, \varphi^i_{\theta^{\alpha}} \;-\; F^i(\boldsymbol{\varphi},\, k_1^{\beta_1}\, \boldsymbol{\varphi}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d}\, \boldsymbol{\varphi}_{\theta^{\beta_d}}, \dots)\right) \frac{d^m\theta}{(2\pi)^m} \;\equiv\; 0$$

for any translationally invariant integral of (2.5). Taking the variation derivative of this relation with respect to $\varphi^j(\boldsymbol{\theta}')$ on Λ we get the required statement.

Thus, we can write

$$\zeta^{(\nu)}_{i[\mathbf{U}]}(\boldsymbol{\theta}) \;=\; \sum_{q=1}^{m+s} c^{\nu}_q(\mathbf{U})\; \kappa^{(q)}_{i[\mathbf{U}]}(\boldsymbol{\theta}) \qquad , \qquad \nu = 1, \dots, N \tag{2.11}$$

with some smooth functions $c^{\nu}_q(\mathbf{U})$ on a complete regular family Λ.

For our consideration of the regular Whitham system we will need a sufficient number of the first integrals (2.8), such that the values of the functionals J^{ν} on Λ represent the full set of parameters

$U^\nu = J^\nu|_\Lambda$. Thus, we will require here the presence of $N = m(d+1)+s$ independent integrals I^ν, $\nu = 1, \dots, m(d+1)+s$. Besides that, we will require that the maximal linearly independent subset of the functions (2.10) gives a complete set of linearly independent left eigenvectors of the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}$ with zero eigenvalues among the vectors regularly depending on the parameters $\mathbf{U}$ on the family Λ.

Coming back to the definition of a complete regular family of m-phase solutions of system (2.5) we can see that in the case of a complete regular family Λ the number of linearly independent vectors (2.10) on Λ is always finite. More precisely, if $N = m(d+1)+s$ is the number of parameters of m-phase solutions of (2.5) (excluding the initial phase shifts) then for a complete regular family of m-phase solutions we require the presence of exactly $m+s = N - md$ left eigenvectors $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0)$ with zero eigenvalues, regularly depending on parameters (in accordance with the number of the vectors $\mathbf{\Phi}_{\theta^\alpha}$, $\mathbf{\Phi}_{n^l}$). Thus, according to Definition 1.1, we assume here that the number of linearly independent vectors defined by formula (2.10) is exactly equal to $m+s = N-md$ for a complete regular family Λ.

We should note that the conditions on the variation derivatives of J^ν formulated above do not contradict to the condition that the values J^ν ($\nu = 1, \dots, N$) can be chosen as parameters U^ν on the family of m-phase solutions. Indeed, the definition of J^ν (2.9) explicitly includes the additional md functions k^α_p, which provide the necessary functional independence of the values of J^ν on Λ. In other words, we can use the Euler - Lagrange expressions for the variation derivatives of I^ν only on subspaces with fixed quasiperiods $(\mathbf{k}_1, \dots, \mathbf{k}_d)$. The variation of the quasiperiods gives linearly growing variations which do not allow to use the Euler - Lagrange expressions.

Moreover, under the assumptions formulated above, we can show that the condition of the completeness of the variation derivatives (2.10) of the functionals J^ν in the space of regular left eigenvectors of the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}$ with zero eigenvalues follows in fact from the condition that the values $U^\nu = J^\nu|_\Lambda$ can be chosen as the full set of parameters (excluding the initial phase shifts) on the family Λ.

Let us make the agreement that we will always assume here that the Jacobian of the coordinate transformation

$$(\mathbf{k}_q,\, \boldsymbol{\omega},\, \mathbf{n}) \;\rightarrow\; \left(U^1,\, \dots,\, U^N\right)$$

is different from zero on Λ whenever we say that the values $U^\nu(\mathbf{k}_q, \boldsymbol{\omega}, \mathbf{n})$ represent a complete set of parameters on Λ (excluding the initial phase shifts).

Under the conditions formulated above let us prove here the following proposition:

Proposition 2.1.

Let Λ be a complete regular family of m-phase solutions of system (2.5). Let the values $(U^1, \dots, U^N)$ of the functionals $(J^1, \dots, J^N)$ (2.9) give a complete set of parameters on Λ excluding the initial phase shifts. Then:

1) The set of the vectors

$$\{\mathbf{\Phi}_{\omega^\alpha}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\, \mathbf{k}_q,\, \boldsymbol{\omega},\, \mathbf{n})\,,\;\; \mathbf{\Phi}_{n^l}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\, \mathbf{k}_q,\, \boldsymbol{\omega},\, \mathbf{n})\,, \qquad \alpha = 1, \dots, m,\;\; l = 1, \dots, s\}$$

is linearly independent on Λ;

2) The variation derivatives $\zeta^{(\nu)}_{i[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0)$, given by (2.10), generate the full space of the regular left eigenvectors of the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}$ with zero eigenvalues on the family Λ.

Proof.

Indeed, we require that the rows given by the derivatives

$$\left({\partial U^{1} \over \partial \omega^{\alpha}}, \dots , {\partial U^{N} \over \partial \omega^{\alpha}} \right) \quad , \quad \left({\partial U^{1} \over \partial n^{l}}, \dots , {\partial U^{N} \over \partial n^{l}} \right)$$

are linearly independent on Λ. Using the expressions

$$\begin{aligned} {\partial U^{\nu} \over \partial \omega^{\alpha}} \,\, &= \,\, \int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi} \zeta^{(\nu)}_{i[\mathbf{U}]}(\boldsymbol{\theta}) \,\, \Phi^{i}_{\omega^{\alpha}}(\boldsymbol{\theta}, \, \mathbf{U}) \,\, {d^{m}\theta \over (2\pi)^{m}} \quad , \qquad \alpha = 1, \dots, m \\ {\partial U^{\nu} \over \partial n^{l}} \,\, &= \,\, \int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi} \zeta^{(\nu)}_{i[\mathbf{U}]}(\boldsymbol{\theta}) \,\, \Phi^{i}_{n^{l}}(\boldsymbol{\theta}, \, \mathbf{U}) \,\, {d^{m}\theta \over (2\pi)^{m}} \quad , \qquad l = 1, \dots, s \end{aligned}$$

on Λ, we get that the set $\{\mathbf{\Phi}_{\omega^{\alpha}}, \, \mathbf{\Phi}_{n^{l}}\}$ is linearly independent on Λ and the number of linearly independent variation derivatives (2.10) is not less than $m + s$.

We obtain then that the variation derivatives (2.10) generate in this case a space of regular left eigenvectors of the operator $\hat{L}^{i}_{j[\mathbf{U},\boldsymbol{\theta}_{0}]}$ with zero eigenvalues of dimension $(m+s)$.

Proposition 2.1 is proved.

Let us prove here the following lemma, which we will need in further considerations.

Lemma 2.1.

Let the values U^{ν} of the functionals J^{ν} on a complete regular family of m-phase solutions Λ be functionally independent and give a complete set of parameters (excluding initial phase shifts) on Λ, such that we have $k^{\alpha}_{p} = k^{\alpha}_{p}(U^{1}, \dots, U^{N})$. Then the functionals $k^{\alpha}_{p}(J^{1}, \dots, J^{N})$ have zero variation derivatives on Λ.

Proof.

As we have seen, the conditions of the Lemma imply the existence of md independent relations

$$\sum_{\nu=1}^{N} \lambda^{\tau}_{\nu}(\mathbf{U}) \,\, {\delta J^{\nu} \over \delta \varphi^{i}(\boldsymbol{\theta})} \Bigg|_{\boldsymbol{\varphi}(\boldsymbol{\theta}) \, = \, \mathbf{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_{0}, \mathbf{U})} \,\, \equiv \,\, 0 \quad , \quad \tau = 1, \dots, md \qquad (2.12)$$

on Λ.

For the corresponding coordinates U^{ν} on Λ this implies the relations

$$\sum_{\nu=1}^{N} \lambda^{\tau}_{\nu}(\mathbf{U}) \, dU^{\nu} \,\, = \,\, \sum_{p=1}^{d} \sum_{\beta=1}^{m} \mu^{(\tau)}_{(\beta p)}(\mathbf{U}) \, dk^{\beta}_{p}(\mathbf{U})$$

for some matrix $\mu^{(\tau)}_{(\beta p)}(\mathbf{U})$.

Let us consider the matrix $\mu^{(\tau)}_{(\beta p)}(\mathbf{U})$ as a $md \times md$ matrix giving a linear transformation $\hat{\mu} : \mathbb{R}^{md} \rightarrow \mathbb{R}^{md}$ between the spaces with bases parametrized by the pairs (βp) and the index τ respectively. Since U^{ν} provide coordinates on Λ the matrix $\mu^{(\tau)}_{(\beta p)}(\mathbf{U})$ has the full rank and, therefore, invertible. We can then write

$$dk^{\beta}_{p} \,\, = \,\, \sum_{\tau=1}^{md} (\hat{\mu}^{-1})^{(\beta p)}_{(\tau)}(\mathbf{U}) \,\, \sum_{\nu=1}^{N} \lambda^{(\tau)}_{\nu}(\mathbf{U}) \, dU^{\nu}$$

The assertion of the Lemma follows then from (2.12).

Lemma 2.1 is proved.

Let us consider now the system

$$\langle P^{\nu} \rangle_{T} \;=\; \langle Q^{\nu 1} \rangle_{X^{1}} \,+\, \dots \,+\, \langle Q^{\nu d} \rangle_{X^{d}} \quad , \quad \nu = 1, \dots, N = m(d+1) + s \tag{2.13}$$

on the space of the functions $\mathbf{U}(\mathbf{X})$, where $\langle \dots \rangle$ denotes the averaging operation on Λ defined by the formula

$$\langle f(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots) \rangle \;\equiv\; \int_{0}^{2\pi} \!\!\dots \int_{0}^{2\pi} f\left(\mathbf{\Phi},\, k_{1}^{\beta_{1}} \mathbf{\Phi}_{\theta^{\beta_{1}}}, \dots, k_{d}^{\beta_{d}} \mathbf{\Phi}_{\theta^{\beta_{d}}}, \dots \right) \frac{d^{m}\theta}{(2\pi)^{m}}$$

Let us prove here the following lemma about the connection between systems (2.13) and (1.11).

Lemma 2.2.

Let the values U^{ν} of the functionals J^{ν} on a complete regular family of m-phase solutions Λ be functionally independent and give a complete set of parameters on Λ excluding the initial phase shifts. Then on the space of functions $\mathbf{U}(\mathbf{X})$ satisfying the system of constraints

$$\frac{\partial k_{p}^{\alpha}(\mathbf{U}(\mathbf{X}))}{\partial X^{l}} \;=\; \frac{\partial k_{l}^{\alpha}(\mathbf{U}(\mathbf{X}))}{\partial X^{p}} \tag{2.14}$$

system (2.13) is equivalent to the evolutionary part (1.11) of the regular Whitham system.

Proof.

Let us introduce the functions

$$\Pi_{i}^{\nu (l_{1} \dots l_{d})} (\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots) \;\equiv\; \frac{\partial P^{\nu}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots)}{\partial \varphi^{i}_{l_{1} x^{1} \dots l_{d} x^{d}}} \quad , \quad l_{1}, \dots, l_{d} \,\geq\, 0 \tag{2.15}$$

Using the expression for the evolution of the densities $P^{\nu}(\boldsymbol{\varphi}, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \dots)$ we can write the following identities

$$P^{\nu}_{t}(\boldsymbol{\varphi}, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \;=\; \sum_{l_{1}, \dots, l_{d}} \epsilon^{l_{1} + \dots + l_{d}} \, \Pi_{i}^{\nu (l_{1} \dots l_{d})} (\boldsymbol{\varphi}, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \, \left(F^{i}(\boldsymbol{\varphi}, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \right)_{l_{1} X^{1} \dots l_{d} X^{d}} \;\equiv$$

$$\equiv\; \epsilon \, Q^{\nu 1}_{X^{1}} (\boldsymbol{\varphi}, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \,+\, \dots \,+\, \epsilon \, Q^{\nu d}_{X^{d}} (\boldsymbol{\varphi}, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \tag{2.16}$$

To calculate the values $\epsilon \langle Q^{\nu 1} \rangle_{X^{1}} \,+\, \dots \,+\, \epsilon \langle Q^{\nu d} \rangle_{X^{d}}$ let us put now

$$\varphi^{i}(\boldsymbol{\theta}, \mathbf{X}) \;=\; \Phi^{i} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right) \tag{2.17}$$

where $S^{\alpha}_{X^{p}} = k^{\alpha}_{p}(\mathbf{U}(\mathbf{X}))$.

The operators $\epsilon \, \partial / \partial X^{p}$ acting on the functions (2.17) can be naturally represented as a sum of $k^{\alpha}_{p} \, \partial / \partial \theta^{\alpha}$ and the terms proportional to ϵ. So, any expression $f(\boldsymbol{\varphi}, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \dots)$ on the submanifold (2.17) can be naturally represented in the form

$$f(\boldsymbol{\varphi}, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \;=\; \sum_{l \geq 0} \epsilon^{l} \, f_{[l]} \, [\mathbf{\Phi}, \mathbf{U}]$$

where $f_{[l]}[\mathbf{\Phi}, \mathbf{U}]$ are smooth functions of $(\mathbf{\Phi}, \mathbf{\Phi}_{\theta^\alpha}, \mathbf{\Phi}_{U^\nu}, \dots)$ and $(\mathbf{U}, \mathbf{U}_{\mathbf{X}}, \mathbf{U}_{\mathbf{XX}}, \dots)$, polynomial in the derivatives $(\mathbf{U}_{\mathbf{X}}, \mathbf{U}_{\mathbf{XX}}, \dots)$, and having degree l in terms of the total number of derivations of $\mathbf{U}$ w.r.t. $\mathbf{X}$. Note also that the functions $\mathbf{\Phi}$ appear in $f_{[l]}$ with the phase shift $\mathbf{S}(\mathbf{X})/\epsilon$ according to (2.17). The common phase shift is not important for the integration with respect to $\boldsymbol{\theta}$, so let us assume below that the phase shift $\mathbf{S}(\mathbf{X})/\epsilon$ is omitted after taking all the differentiations with respect to $\mathbf{X}$.

According to (2.16) and (2.6) we can write

$$\epsilon \, \langle Q^{\nu 1} \rangle_{X^1} \; + \; \dots \; + \; \epsilon \, \langle Q^{\nu d} \rangle_{X^d} \;\; = \;\; \epsilon \int_0^{2\pi} \!\! \dots \int_0^{2\pi} \left(Q^{\nu 1}_{X^1 [1]} \; + \; \dots \; + \; Q^{\nu d}_{X^d [1]} \right) \frac{d^m \theta}{(2\pi)^m} \;\; =$$

$$= \;\; \epsilon \int_0^{2\pi} \!\! \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \left(\Pi^{\nu (l_1 \dots l_d)}_{i \, [0]} \, F^i_{l_1 X^1 \dots l_d X^d \, [1]} \; + \; \Pi^{\nu (l_1 \dots l_d)}_{i \, [1]} \, F^i_{l_1 X^1 \dots l_d X^d \, [0]} \right) \frac{d^m \theta}{(2\pi)^m} \;\; =$$

$$= \;\; \epsilon \int_0^{2\pi} \!\! \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \Bigg(\Pi^{\nu (l_1 \dots l_d)}_{i \, [0]} \;\, k_1^{\gamma^1_1} \dots k_1^{\gamma^1_{l_1}} \;\; \dots \;\; k_d^{\gamma^d_1} \dots k_d^{\gamma^d_{l_d}} \;\; F^i_{[1] \, \theta^{\gamma^1_1} \dots \theta^{\gamma^1_{l_1}} \dots \theta^{\gamma^d_1} \dots \theta^{\gamma^d_{l_d}}} \;\; +$$

$$+ \;\; \Pi^{\nu (l_1 \dots l_d)}_{i \, [0]} \;\, \left(\omega^\beta \, \Phi^i_{\theta^\beta} \right)_{l_1 X^1 \dots l_d X^d \, [1]} \;\; + \;\; \Pi^{\nu (l_1 \dots l_d)}_{i \, [1]} \;\, \left(\omega^\beta \, \Phi^i_{\theta^\beta} \right)_{l_1 X^1 \dots l_d X^d \, [0]} \Bigg) \frac{d^m \theta}{(2\pi)^m} \;\; =$$

$$= \;\; \epsilon \int_0^{2\pi} \!\! \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \Bigg(k_1^{\gamma^1_1} \dots k_1^{\gamma^1_{l_1}} \;\; \dots \;\; k_d^{\gamma^d_1} \dots k_d^{\gamma^d_{l_d}} \;\, (-1)^{l_1 + \dots + l_d} \;\, \Pi^{\nu (l_1 \dots l_d)}_{i \, [0] \, \theta^{\gamma^1_1} \dots \theta^{\gamma^1_{l_1}} \dots \theta^{\gamma^d_1} \dots \theta^{\gamma^d_{l_d}}} \;\; F^i_{[1]} \;\; +$$

$$+ \;\; \omega^\beta_{X^1} \; \Pi^{\nu (l_1 \dots l_d)}_{i [0]} \;\, l_1 \; \Phi^i_{\theta^\beta \, (l_1 - 1) X^1 \dots l_d X^d \, [0]} \;\; + \;\; \omega^\beta_{X^d} \; \Pi^{\nu (l_1 \dots l_d)}_{i [0]} \;\, l_d \; \Phi^i_{\theta^\beta \, l_1 X^1 \dots (l_d - 1) X^d \, [0]} \;\; +$$

$$+ \;\; \omega^\beta \; \Pi^{\nu (l_1 \dots l_d)}_{i [0]} \;\, \Phi^i_{\theta^\beta \, l_1 X^1 \dots l_d X^d \, [1]} \;\; + \;\; \omega^\beta \; \Pi^{\nu (l_1 \dots l_d)}_{i [1]} \;\, \Phi^i_{\theta^\beta \, l_1 X^1 \dots l_d X^d \, [0]} \Bigg) \frac{d^m \theta}{(2\pi)^m}$$

The last two terms in the above expression represent the integral of the value

$$\omega^\beta \sum_{l_1, \dots, l_d} \left(\Pi^{\nu (l_1 \dots l_d)}_i \, \Phi^i_{\theta^\beta \, l_1 X^1 \dots l_d X^d} \right)_{[1]} \;\; \equiv \;\; \omega^\beta \, \partial P^\nu_{[1]} / \partial \theta^\beta$$

and are equal to zero.

It is not difficult to see also that for arbitrary dependence of parameters $\mathbf{U}$ of T, the derivative of the average $\langle P^\nu \rangle$ w.r.t. T can be written as:

$$\langle P^\nu \rangle_T \;\; = \;\; \int_0^{2\pi} \!\! \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \Pi^{\nu (l_1 \dots l_d)}_{i \, [0]} \; \left(k_1^{\gamma^1_1} \dots k_1^{\gamma^1_{l_1}} \;\; \dots \;\; k_d^{\gamma^d_1} \dots k_d^{\gamma^d_{l_d}} \;\; \Phi^i_{\theta^{\gamma^1_1} \dots \theta^{\gamma^1_{l_1}} \dots \theta^{\gamma^d_1} \dots \theta^{\gamma^d_{l_d}}} \right)_T \frac{d^m \theta}{(2\pi)^m}$$

Now, we can write the relations $\langle P^\nu \rangle_T \; - \; \langle Q^{\nu 1} \rangle_{X^1} - \dots - \langle Q^{\nu d} \rangle_{X^d} \; = \; 0$ as

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \Big(\zeta^{(\nu)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta}) \; \big[\Phi^i_T(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X})) \; - \; F^i_{[1]}(\boldsymbol{\theta}, \mathbf{X}) \big] \; +$$

$$+ \; (k^\beta_{1T} \, - \, \omega^\beta_{X^1}) \sum_{l_1,\dots,l_d} \Pi^{\nu (l_1 \dots l_d)}_{i\,[0]} \; l_1 \; k_1^{\gamma^1_1} \dots k_1^{\gamma^1_{l_1-1}} \; \dots \; k_d^{\gamma^d_1} \dots k_d^{\gamma^d_{l_d}} \;\; \Phi^i_{\theta^\beta \theta^{\gamma^1_1} \dots \theta^{\gamma^1_{l_1-1}} \dots \theta^{\gamma^d_1} \dots \theta^{\gamma^d_{l_d}}} \; + \; \dots$$

$$+ \; (k^\beta_{dT} \, - \, \omega^\beta_{X^d}) \sum_{l_1,\dots,l_d} \Pi^{\nu (l_1 \dots l_d)}_{i\,[0]} \; l_d \; k_1^{\gamma^1_1} \dots k_1^{\gamma^1_{l_1}} \; \dots \; k_d^{\gamma^d_1} \dots k_d^{\gamma^d_{l_d-1}} \;\; \Phi^i_{\theta^\beta \theta^{\gamma^1_1} \dots \theta^{\gamma^1_{l_1}} \dots \theta^{\gamma^d_1} \dots \theta^{\gamma^d_{l_d-1}}} \Big) \; \frac{d^m\theta}{(2\pi)^m} \;\; = \;\; 0$$

where the values $\zeta^{(\nu)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta})$ are given by (2.10).

Consider the convolution (in ν) of the above expression with the values $\partial k^\alpha_p / \partial U^\nu$. The expressions

$$\frac{\partial k^\alpha_p}{\partial U^\nu} (\mathbf{U}(\mathbf{X})) \;\; \zeta^{(\nu)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta})$$

are identically equal to zero according to Lemma 2.1.

From the other hand we have

$$\frac{\partial k^\alpha_p}{\partial U^\nu} \int_0^{2\pi}\!\!\dots\int_0^{2\pi} \sum_{l_1,\dots,l_d} l_q \, k_1^{\gamma^1_1} \dots k_1^{\gamma^1_{l_1}} \; \dots \; k_q^{\gamma^q_1} \dots k_q^{\gamma^q_{l_q-1}} \; \dots \; k_d^{\gamma^d_1} \dots k_d^{\gamma^d_{l_d}} \; \times$$

$$\times \; \Phi^i_{\theta^\beta \theta^{\gamma^1_1} \dots \theta^{\gamma^1_{l_1}} \dots \theta^{\gamma^q_1} \dots \theta^{\gamma^q_{l_q-1}} \dots \theta^{\gamma^d_1} \dots \theta^{\gamma^d_{l_d}}} \quad \Pi^{\nu (l_1 \dots l_d)}_{i\,[0]} \quad \frac{d^m\theta}{(2\pi)^m} \;\; =$$

$$= \;\; \left(\frac{\partial k^\alpha_p}{\partial U^\nu} \, \frac{\partial}{\partial k^\beta_q} \, J^\nu [\boldsymbol{\varphi}, \mathbf{k}_1, \dots, \mathbf{k}_d] \right) \Bigg|_{\boldsymbol{\varphi}(\boldsymbol{\theta}) = \boldsymbol{\Phi}(\boldsymbol{\theta}, \mathbf{U})} \;\; = \;\; \delta^\alpha_\beta \; \delta^q_p \qquad (2.18)$$

since the variations of the functions $\boldsymbol{\Phi}$ are insignificant for the values of k^α_p according to Lemma 2.1.

We get then that conditions (2.13) imply the relations $k^\alpha_{pT} = \omega^\alpha_{X^p}$, which are the second part of system (1.11).

Now the conditions

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \zeta^{(\nu)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta}) \; \big[\Phi^i_T(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X})) \; - \; F^i_{[1]}(\boldsymbol{\theta}, \mathbf{X}) \big] \; \frac{d^m\theta}{(2\pi)^m} \;\; = \;\; 0$$

express the conditions of orthogonality of the vectors (2.10) to the function $-\boldsymbol{\Phi}_T + \mathbf{F}_{[1]}$, which coincides exactly with the right-hand part of equation (1.7) in our case. Since the linear span of the vectors (2.10) coincides with the linear span of the complete set of the regular left eigenvectors of the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(\mathbf{X}, T)$ with zero eigenvalues, we get that system (2.13) is equivalent to system (1.11) under the constraints (2.14).

Lemma 2.2 is proved.

Let us note here that it follows from Lemma 2.2 that systems (2.13), obtained from different sets of conservation laws are equivalent to each other on the submanifold given by constraints (2.14). In

other words, if system (2.5) has additional conservation laws of the form (2.8) then their averaging gives relations following from system (2.13) under the additional constraints (2.14).

The Hamiltonian properties of systems (2.13) and more general systems (1.13) are very well developed in the one-dimensional situation. The general theory of the one-dimensional systems (1.13), which are Hamiltonian with respect to local Poisson brackets of Hydrodynamic type (Dubrovin - Novikov brackets) was constructed by B.A. Dubrovin and S.P. Novikov. Let us give here a brief description of the Dubrovin - Novikov Hamiltonian structures and of the properties of the corresponding systems (1.13).

The Dubrovin - Novikov bracket on the space of fields $(U^{1}(X), \dots, U^{N}(X))$ has the form

$$\{U^{\nu}(X), U^{\mu}(Y)\} \; = \; g^{\nu\mu}(\mathbf{U})\, \delta'(X-Y) \, + \, b^{\nu\mu}_{\gamma}(\mathbf{U})\, U^{\gamma}_{X}\, \delta(X-Y) \quad , \quad \nu, \mu = 1, \dots, N \tag{2.19}$$

The Hamiltonian operator corresponding to (2.19) can be written in the form

$$\hat{J}^{\nu\mu} \; = \; g^{\nu\mu}(\mathbf{U}) \frac{d}{dX} \, + \, b^{\nu\mu}_{\gamma}(\mathbf{U})\, U^{\gamma}_{X} \tag{2.20}$$

As was shown by B.A. Dubrovin and S.P. Novikov ([9, 11, 12]), the expression (2.19) with non-degenerate tensor $g^{\nu\mu}(\mathbf{U})$ defines a Poisson bracket on the space of fields $\mathbf{U}(X)$ if and only if:

1) Tensor $g^{\nu\mu}(\mathbf{U})$ gives a symmetric flat pseudo-Riemannian metric with upper indexes on the space of parameters $(U^{1}, \dots, U^{N})$;

2) The values

$$\Gamma^{\nu}_{\mu\gamma} = -g_{\mu\lambda} \; b^{\lambda\nu}_{\gamma}$$

where $g^{\nu\lambda}(\mathbf{U})\, g_{\lambda\mu}(\mathbf{U}) = \delta^{\nu}_{\mu}$, represent the Christoffel symbols for the corresponding metric $g_{\nu\mu}(\mathbf{U})$.

As follows from the statements above, every Dubrovin - Novikov bracket with non-degenerate tensor $g^{\nu\mu}(\mathbf{U})$ can be written in the canonical form ([9, 11, 12]):

$$\{n^{\nu}(X), n^{\mu}(Y)\} \; = \; \epsilon^{\nu} \delta^{\nu\mu}\, \delta'(X-Y) \quad , \quad \epsilon^{\nu} = \pm 1$$

after the transition to the flat coordinates $n^{\nu} = n^{\nu}(\mathbf{U})$ for the metric $g_{\nu\mu}(\mathbf{U})$.

The functionals

$$N^{\nu} \; = \; \int_{-\infty}^{+\infty} n^{\nu}(X)\, dX$$

represent the annihilators of the Dubrovin - Novikov bracket while the functional

$$P \; = \; \int_{-\infty}^{+\infty} \frac{1}{2} \sum_{\nu=1}^{N} \epsilon^{\nu} (n^{\nu})^{2}(X)\, dX$$

represents the momentum functional for the bracket (2.19).

The Hamiltonian functions in the theory of brackets (2.19) are represented by the functionals of Hydrodynamic Type, i.e.

$$H \;=\; \int_{-\infty}^{+\infty} h(\mathbf{U})\, dX \tag{2.21}$$

The bracket (2.19) has also two other important forms on the space of $\mathbf{U}(X)$. One of them is the "Liouville" form ([9, 11, 12]) having the form

$$\{U^{\nu}(X), U^{\mu}(Y)\} \;=\; (\gamma^{\nu\mu}(\mathbf{U}) \,+\, \gamma^{\mu\nu}(\mathbf{U}))\, \delta'(X-Y) \,+\, {\partial \gamma^{\nu\mu} \over \partial U^{\lambda}}\, U^{\lambda}_{X}\; \delta(X-Y)$$

for some functions $\gamma^{\nu\mu}(\mathbf{U})$.

The "Liouville" form of the Dubrovin - Novikov bracket is called also the physical form and corresponds to the case when the integrals of coordinates U^{ν}

$$I^{\nu} \;=\; \int_{-\infty}^{+\infty} U^{\nu}(X)\, dX$$

commute with each other.

Another important form of the Dubrovin - Novikov bracket is the diagonal form. It corresponds to the case when the coordinates U^{ν} represent the diagonal coordinates for the metric $g_{\nu\mu}(\mathbf{U})$ and the tensor $g^{\nu\mu}(\mathbf{U})$ in (2.19) has a diagonal form. This form of the Dubrovin - Novikov bracket is closely connected with the integration theory of systems of Hydrodynamic Type

$$U^{\nu}_{T} \;=\; V^{\nu}_{\mu}(\mathbf{U})\, U^{\mu}_{X} \tag{2.22}$$

which can be written in the diagonal form

$$U^{\nu}_{T} \;=\; V^{\nu}(\mathbf{U})\, U^{\nu}_{X} \tag{2.23}$$

(no summation) and are Hamiltonian with respect to the bracket (2.19).

It was conjectured by S.P. Novikov that all the systems of Hydrodynamic Type having the form (2.23) and Hamiltonian with respect to any bracket (2.19) are integrable. This conjecture was proved by S.P. Tsarev in [34] where the method (the "generalized hodograph method") of integration of these systems was suggested. However, the method of Tsarev proved to be applicable to a wider class of diagonalizable systems of hydrodynamic type which was called by Tsarev "semi-Hamiltonian". As it turned out later in the class of "semi-Hamiltonian systems" fall also the systems Hamiltonian with respect to generalizations of the Dubrovin - Novikov bracket - the weakly nonlocal Mokhov - Ferapontov bracket ([26]) and the Ferapontov bracket ([13, 14]). Various aspects of the weakly nonlocal brackets of Hydrodynamic Type are discussed in [26, 13, 14, 15, 16, 32, 24].

Let us describe also the procedure for constructing the Dubrovin - Novikov bracket in the one-dimensional case when the original system (2.5) is Hamiltonian with respect to a local field-theoretic bracket

$$\{\varphi^{i}(x), \varphi^{j}(y)\} \;=\; \sum_{k\geq 0} B^{ij}_{(k)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{x}, \dots)\, \delta^{(k)}(x-y) \tag{2.24}$$

with the local Hamiltonian of the form

$$H \;=\; \int P_{H}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{x}, \dots)\, dx \tag{2.25}$$

which was suggested by B.A. Dubrovin and S.P. Novikov in [9, 11, 12].

Method of B.A. Dubrovin and S.P. Novikov is based on the existence of N (equal to the number of parameters U^{ν} of the family of m-phase solutions of (2.5)) local integrals

$$I^{\nu} \;=\; \int P^{\nu}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{x}, \dots)\, dx \tag{2.26}$$

which commute with Hamiltonian (2.25) and with each other

$$\{I^{\nu}, H\} = 0 \quad , \quad \{I^{\nu}, I^{\mu}\} = 0 \tag{2.27}$$

and can be described as follows:

We calculate the pairwise Poisson brackets of the densities P^{ν} having the form

$$\{P^{\nu}(x), P^{\mu}(y)\} \;=\; \sum_{k\geq 0} A^{\nu\mu}_{k}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{x}, \dots)\, \delta^{(k)}(x-y) \tag{2.28}$$

where

$$A^{\nu\mu}_{0}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{x}, \dots) \;\equiv\; \partial_{x} Q^{\nu\mu}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{x}, \dots) \tag{2.29}$$

according to (2.27).

The corresponding Dubrovin-Novikov bracket on the space of functions $\mathbf{U}(X)$ has the form:

$$\{U^{\nu}(X), U^{\mu}(Y)\} \;=\; \langle A^{\nu\mu}_{1}\rangle(\mathbf{U})\; \delta'(X-Y) \;+\; \frac{\partial \langle Q^{\nu\mu}\rangle}{\partial U^{\gamma}}\; U^{\gamma}_{X}\; \delta(X-Y) \tag{2.30}$$

Let us remind that we assume that the parameters U^{ν} coincide with the values of the functionals I^{ν} defined on the corresponding quasiperiodic solutions of the family Λ:

$$U^{\nu} = \langle P^{\nu}(x)\rangle$$

The Whitham system

$$\langle P^{\nu}\rangle_{T} \;=\; \langle Q^{\nu}\rangle_{X} \quad , \qquad \nu = 1, \dots, N$$

is Hamiltonian with respect to the Dubrovin - Novikov bracket (2.30) with the Hamiltonian

$$H^{av} \;=\; \int_{-\infty}^{+\infty} \langle P_{H}\rangle\, (\mathbf{U}(X))\;\; dX \tag{2.31}$$

The proof of the Jacobi identity for the bracket (2.30) was suggested in [22] under certain assumptions about the family of m-phase solutions of (2.5). Besides that, it was shown in [21] that the Dubrovin - Novikov procedure is compatible with the procedure of averaging of local Lagrangian functions when carrying out of both the procedures is possible. Let us note also that the generalization of the Dubrovin - Novikov procedure for the weakly nonlocal case was proposed in [23].

In paper [33] all the local brackets (2.19) for the Whitham equations for KdV, NLS, and SG equations were found. Besides that, in papers [2, 3] the hierarchies of the weakly nonlocal Hamiltonian structures for the Whitham systems for KdV were represented.

The most detailed discussion and justification of the Dubrovin - Novikov procedure separately for the single-phase and the multiphase cases can be found in [25]. In particular, it was shown that

the justification of the procedure is in fact insensitive to the appearance of "resonances" which can arise in the multi-phase case, which is the basis for its widespread use in the multiphase situation.

In this paper we are going to consider multi-dimensional evolution systems (2.5) which are Hamiltonian with respect to a local field-theoretic Poisson bracket

$$\{\varphi^i(\mathbf{x})\,,\,\varphi^i(\mathbf{y})\} \;=\; \sum_{l_1,\dots,l_d} B^{ij}_{(l_1,\dots,l_d)}(\boldsymbol{\varphi},\boldsymbol{\varphi}_{\mathbf{x}},\dots)\;\delta^{(l_1)}(x^1-y^1)\;\dots\;\delta^{(l_d)}(x^d-y^d) \tag{2.32}$$

$(l_1,\dots,l_d \geq 0)$, with a local Hamiltonian of the form

$$H \;=\; \int P_H\left(\boldsymbol{\varphi},\boldsymbol{\varphi}_{\mathbf{x}},\boldsymbol{\varphi}_{\mathbf{xx}},\dots\right)\; d^d x \tag{2.33}$$

As we said already, we are going to consider complete regular families Λ of m-phase solutions of system (2.5) satisfying system (2.6) with some values of $(k^\alpha_p(\mathbf{U}),\,\omega^\alpha(\mathbf{U}))$. We assume here that the solutions of the family Λ are parametrized by $m(d+1)+s$ independent parameters $(k^\alpha_p,\omega^\alpha,n^l)$, $\alpha=1,\dots,m,\ p=1,\dots,d,\ l=1,\dots,s$ excluding initial phase shifts θ^α_0, and we have a set of $m(d+1)+s$ independent first integrals I^ν of the form (2.8) such that their values on Λ can be chosen as the coordinate system $\{U^\nu\}$ on Λ (excluding initial phase shifts). We will assume also, that the integrals I^ν commute with each other and with the Hamiltonian H

$$\{I^\nu\,,\,I^\mu\} \;=\; 0 \quad , \quad \{I^\nu\,,\,H\} \;=\; 0 \tag{2.34}$$

according to bracket (2.32).

Let us note that according to (2.34) the flows

$$\varphi^i_{t^\nu} \;=\; S^{i\nu}(\boldsymbol{\varphi},\boldsymbol{\varphi}_{\mathbf{x}},\dots) \;=\; \{\varphi^i(\mathbf{x})\,,\,I^\nu\} \tag{2.35}$$

generated by the functionals I^ν according to bracket (2.32) commute with the initial flow (2.5). The flows (2.35) leave invariant the full families of m-phase solutions of (2.5) as well as the values $U^\nu=I^\nu$ of the functionals I^ν on them. For a complete regular family of m-phase solutions with independent parameters $(k^1_p,\dots,k^m_p)$ it's not difficult to show that the flows (2.35) generate linear (in time) shifts of the phases θ^α_0 with some constant frequencies $\omega^{\alpha\nu}(\mathbf{U})$, such that

$$S^{i\nu}\left(\boldsymbol{\Phi},\,k^{\beta_1}_1\,\boldsymbol{\Phi}_{\theta^{\beta_1}},\dots\,k^{\beta_d}_d\,\boldsymbol{\Phi}_{\theta^{\beta_d}},\dots\right) \;=\; \omega^{\alpha\nu}(\mathbf{U})\,\Phi^i_{\theta^\alpha}(\boldsymbol{\theta},\mathbf{U}) \tag{2.36}$$

According to Lemma 2.1 we have also that the functionals $k^\alpha_p(\mathbf{I})$ should generate the zero flows on the corresponding family of m-phase solutions of (2.5). For U^ν coinciding with the values of I^ν on Λ we get then the relations

$$\frac{\partial k^\alpha_p(\mathbf{U})}{\partial U^\nu}\;\omega^{\beta\nu}(\mathbf{U}) \;\equiv\; 0 \quad , \quad \alpha,\beta \;=\; 1,\dots,m,\ \ p \;=\; 1,\dots,d \tag{2.37}$$

By analogy with the one-dimensional case we can try to construct an analogue of the Dubrovin - Novikov procedure for $d>1$. Let us represent the pairwise Poisson brackets of the densities $P^\nu(\mathbf{x})$, $P^\mu(\mathbf{y})$ in the form

$$\{P^{\nu}(\mathbf{x})\,,\,P^{\mu}(\mathbf{y})\} \;=\; \sum_{l_1,\dots,l_d} A^{\nu\mu}_{l_1\dots l_d}(\boldsymbol{\varphi},\boldsymbol{\varphi}_{\mathbf{x}},\dots)\;\delta^{(l_1)}(x^1-y^1)\;\dots\;\delta^{(l_d)}(x^d-y^d)$$

$(l_1,\dots,l_d \geq 0)$.

According to relations (2.34) we can also write here the relations

$$A^{\nu\mu}_{0\dots0}(\boldsymbol{\varphi},\boldsymbol{\varphi}_{\mathbf{x}},\dots) \;\equiv\; \partial_{X^1}\,Q^{\nu\mu1}(\boldsymbol{\varphi},\boldsymbol{\varphi}_{\mathbf{x}},\dots)\,+\,\dots\,+\,\partial_{X^d}\,Q^{\nu\mu d}(\boldsymbol{\varphi},\boldsymbol{\varphi}_{\mathbf{x}},\dots)$$

for some functions $(Q^{\nu\mu1},\dots,Q^{\nu\mu d})$.

In the full analogy with the Dubrovin - Novikov procedure we can define the expressions

$$\begin{aligned}\{U^{\nu}(\mathbf{X})\,,\,U^{\mu}(\mathbf{Y})\} \;&=\; \langle A^{\nu\mu}_{10\dots0}\rangle(\mathbf{X})\;\delta'(X^1-Y^1)\,\delta(X^2-Y^2)\,\dots\,\delta(X^d-Y^d)\;+\;\dots\\ &+\;\langle A^{\nu\mu}_{0\dots01}\rangle(\mathbf{X})\;\delta(X^1-Y^1)\,\delta(X^2-Y^2)\,\dots\,\delta'(X^d-Y^d)\;+\\ &+\;\big(\langle Q^{\nu\mu1}\rangle_{X^1}\;+\cdots+\;\langle Q^{\nu\mu d}\rangle_{X^d}\big)\;\delta(X^1-Y^1)\,\dots\,\delta(X^d-Y^d)\end{aligned}\tag{2.38}$$

which gives a skew-symmetric (contravariant) form on the space of functions $\mathbf{U}(\mathbf{X})$.

The theory of the Poisson brackets having the form (2.38) in $d>1$ dimensions was considered in [10, 27, 28]. As was shown in [10, 27, 28], the restrictions on the form of (2.38) in $d>1$ dimensions are much stronger than in the one-dimensional case and in general should be considered by the methods of the theory of integrable systems.

However, the procedure described above does not give in general a Poisson bracket on the space of fields $\mathbf{U}(\mathbf{X})$ in the multi-dimensional situation, since the expression (2.38) does not satisfy the Jacobi identity for $d>1$ in general case. Fortunately, we don't need a Poisson bracket on the full space of fields $\mathbf{U}(\mathbf{X})$ for the Hamiltonian formulation of the Whitham approach since the regular Whitham system is defined on the submanifold in the space of $\{\mathbf{U}(\mathbf{X})\}$ given by constraints (2.14). As we will show in the next chapters, expressions (2.38) define indeed a Poisson structure after the restriction on the submanifold mentioned above which gives a Hamiltonian structure for the regular Whitham system for $d>1$.

For the description of the Hamiltonian structure of the regular Whitham system for $d>1$ it is convenient to introduce a coordinate system on the submanifold corresponding to the Whitham solutions. It is most natural to consider then the functions $S^{\alpha}(\mathbf{X})$, $\alpha=1,\dots,m$ such that $S^{\alpha}_{X^p}(\mathbf{X}) \equiv k^{\alpha}_p(\mathbf{X})$ as a part of a coordinate system on the submanifold given by constraints (2.14). However, the $\mathbf{X}$-derivatives of the functions $S^{\alpha}(\mathbf{X})$ give just md independent parameters in the space $\mathbf{U}(\mathbf{X})$ at every given $\mathbf{X}$. To get the full system of coordinates on the submanifold defined by constraints (2.14) we need additional $m+s$ parameters at every point $\mathbf{X}$. It will be convenient for us here to take arbitrary $m+s$ parameters $U^{\gamma}(\mathbf{X})$, $\gamma=1,\dots,m+s$ from the set $\{\mathbf{U}(\mathbf{X})\}$ which are functionally independent with the set $\{k^{\alpha}_p(\mathbf{U})\}$.

Let us note that we put here $\gamma=1,\dots,m+s$ without any loss of generality. As a necessary condition of the functional independence of the set $\{k^{\alpha}_p(\mathbf{U}),U^1,\dots,U^{m+s}\}$ we have, in particular, that the variation derivatives (2.10) of the corresponding functionals $(J^1,\dots,J^{m+s})$ give a linearly independent set on the family Λ. As a consequence, we can claim that the regular left eigen-vectors $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0)$ of the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}$ can be also expressed as linear combinations of the vectors $\boldsymbol{\zeta}^{(\gamma)}_{[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0)$, $\gamma=1,\dots,m+s$, such that we have

$$\kappa^{(q)}_{i[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0) \;=\; \sum_{\gamma=1}^{m+s} d^{(q)}_{\gamma}(\mathbf{U})\; \zeta^{(\gamma)}_{i[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0) \tag{2.39}$$

with some functions $d^{(q)}_{\gamma}(\mathbf{U})$ on Λ.

Representing the frequencies $\omega^{\alpha}(\mathbf{U})$ as functions of the new parameters

$$\omega^{\alpha} \;=\; \omega^{\alpha}\left(\mathbf{S_X},\, U^1, \dots, U^{m+s}\right)$$

we can write the second part of the evolution system (1.11) together with the constraints (1.12) in the form

$$S^{\alpha}_T \;=\; \omega^{\alpha}\left(\mathbf{S_X},\, U^1, \dots, U^{m+s}\right) \tag{2.40}$$

The remaining part of the regular Whitham system can be written as the relations

$$U^{\gamma}_T \;=\; \langle Q^{\gamma 1}\rangle_{X^1} \;+\; \dots \;+\; \langle Q^{\gamma d}\rangle_{X^d} \quad , \qquad \gamma = 1, \dots, m+s \tag{2.41}$$

where $\langle Q^{\gamma p}\rangle \;=\; \langle Q^{\gamma p}\rangle(\mathbf{S_X},\, U^1, \dots, U^{m+s})$.

As follows from our considerations above the choice of the functionally independent parameters U^{γ}, $\gamma = 1, \dots, m+s$ (and the corresponding choice of the functionals I^{γ}) is unessential for the construction.

We will show below, that the regular Whitham system given by equations (2.40) - (2.41) is Hamiltonian with respect to the Poisson bracket given by the relations

$$\{S^{\alpha}(\mathbf{X})\,,\, S^{\beta}(\mathbf{Y})\} \;=\; 0 \quad , \quad \{S^{\alpha}(\mathbf{X})\,,\, U^{\gamma}(\mathbf{Y})\} \;=\; \omega^{\alpha\gamma}\left(\mathbf{S_X},\, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})\right)\; \delta(\mathbf{X}-\mathbf{Y})$$

$\alpha, \beta = 1, \dots, m$, $\gamma = 1, \dots, m+s$, and relations (2.38) for the skew-symmetric form $\{U^{\gamma}(\mathbf{X})\,,\, U^{\rho}(\mathbf{Y})\}$ restricted to the subset $\gamma, \rho = 1, \dots, m+s$, where

$$\langle A^{\gamma\rho}_{0\dots1\dots0}\rangle \;=\; \langle A^{\gamma\rho}_{0\dots1\dots0}\rangle\left(\mathbf{S_X},\, U^1, \dots, U^{m+s}\right) \quad , \quad \langle Q^{\gamma\rho p}\rangle \;=\; \langle Q^{\gamma\rho p}\rangle\left(\mathbf{S_X},\, U^1, \dots, U^{m+s}\right)$$

The Hamiltonian function for the regular Whitham system is also local in this case and is equal to

$$H \;=\; \int \langle P_H\rangle\, \left(\mathbf{S_X},\, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})\right)\; d^d X$$

Let us note here that although system (2.40) - (2.41) as well as the Hamiltonian structure described above can be formally defined with the aid of the $m+s$ additional integrals I^{γ}, the presence of the full set $\{I^{\nu},\, \nu = 1, \dots, m(d+1)+s\}$ giving the full set of parameters $\mathbf{U}$ on the family Λ is important for the justification of the construction of the Hamiltonian structure in our scheme. The rest of equations (2.13) gives in this situation additional conservation laws for the regular Whitham system. Let us note also, that the choice of the functionals $\{I^{\gamma},\, \gamma = 1, \dots, m+s\}$ is also not important for the construction of the Hamiltonian structure of the regular Whitham system as well as for the Whitham system itself.[1]

[1] We mean here that the Hamiltonian structures obtained with the aid of different subsets $\{I^{\gamma},\, \gamma = 1, \dots, m+s\}$ transform into each other after the corresponding change of coordinates.

In the next chapter we start the consideration of the averaging procedure for the bracket (2.32) giving the Hamiltonian structure for the regular Whitham system (2.40) - (2.41).

3 The Dirac restriction and the averaging of the Poisson brackets.

Let us start with the definition of a regular Hamiltonian family of m-phase solutions of system (2.5) and of a complete Hamiltonian set of the functionals (2.8).

Definition 3.1.

We call family Λ of m-phase solutions of system (2.5) a regular Hamiltonian family if:

1) It represents a complete regular family of m-phase solutions of (2.5) in the sense of Definition 1.1;

2) The corresponding bracket (2.32) has on Λ constant number of annihilators N^1, ..., N^s with linearly independent variation derivatives $\delta N^l/\delta\varphi^i(\mathbf{x})$ which coincides with the number of independent annihilators in the neighborhood of Λ.

Let us say that according to the generalized Darboux theorem we can identify the number of the variation derivatives $\delta N^l/\delta\varphi^i(\mathbf{x})$ on Λ with the number of linearly independent quasiperiodic solutions $v_i^{(l)}(\mathbf{x})$ of the equation

$$\sum_{l_1,\dots,l_d} B^{ij}_{(l_1,\dots,l_d)}(\boldsymbol{\varphi},\boldsymbol{\varphi}_{\mathbf{x}},\dots)\Big|_{\Lambda} v^{(l)}_{j,x^{l_1}\dots x^{l_d}}(\mathbf{x}) \;=\; 0$$

where $v_i^{(l)}(\mathbf{x})$ have the same quasiperiods as the corresponding functions $\boldsymbol{\varphi}(\mathbf{x})$ on Λ.

Definition 3.2.

We call a set $(I^1,\dots,I^N)$ of commuting functionals (2.8) a complete Hamiltonian set on a regular Hamiltonian family Λ of m-phase solutions of system (2.5) if:

1) The restriction of the functionals $(I^1,\dots,I^N)$ on the quasiperiodic solutions of the family Λ gives a complete set of parameters $(U^1,\dots,U^N)$ on this family;

2) The Hamiltonian flows generated by $(I^1,\dots,I^N)$ generate on Λ linear phase shifts of $\boldsymbol{\theta}_0$ with frequencies $\boldsymbol{\omega}^\nu(\mathbf{U})$, such that

$$\mathrm{rk}\;||\omega^{\alpha\nu}(\mathbf{U})|| \;=\; m$$

3) The linear space generated by the variation derivatives $\delta I^\nu/\delta\varphi^i(\mathbf{x})$ on Λ contains the variation derivatives of all the annihilators N^q of the bracket (2.32), such that

$$\frac{\delta N^l}{\delta\varphi^i(\mathbf{x})}\Big|_{\Lambda} \;=\; \sum_{\nu=1}^{N}\gamma^l_\nu(\mathbf{U})\;\frac{\delta I^\nu}{\delta\varphi^i(\mathbf{x})}\Big|_{\Lambda}$$

for some smooth functions $\gamma^l_\nu(\mathbf{U})$ on the family Λ.

Let us note that it follows from Definition 3.2 that if a complete Hamiltonian set of integrals $(I^1,\dots,I^N)$ exists for a regular Hamiltonian family Λ then the number of the additional parameters $(n^1,\dots,n^s)$ discussed above is equal to the number of annihilators of the bracket (2.32). Indeed,

according to Definitions 3.1 and 3.2, the number of the functionals I^ν having linearly independent variation derivatives on Λ exactly equals to $m+s$, where s is the number of annihilators of the bracket (2.32). The total number of independent parameters U^ν on Λ is then equal to $m(d+1)+s$ due to the wave vectors k^α_p, $\alpha=1,\dots,m$, $p=1,\dots,d$ which implies the above assertion.

As follows from the condition (2) of Definition 3.2 and from the invariance of the functionals N^l and I^ν with respect to the flows (2.35), the values $\gamma^l_\nu(\mathbf{U})$ can always be chosen independent on the initial phase shifts on the family Λ. The values $\delta I^\nu/\delta\varphi^i(\mathbf{x})|_\Lambda$ are linearly dependent on Λ, so it's natural to choose a complete linearly independent subsystem. Remembering that the variation derivatives of J^ν (2.10) are linear combinations of the regular left eigenvectors $\kappa^{(q)}_{i[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0)$, we can write

$$\left.\frac{\delta N^l}{\delta\varphi^i(\mathbf{x})}\right|_{\boldsymbol{\varphi}(\mathbf{x})=\boldsymbol{\Phi}(\mathbf{k}_p(\mathbf{U})x^p+\boldsymbol{\theta}_0,\mathbf{U})} \;=\; \sum_{q=1}^{m+s} n^l_q(\mathbf{U})\;\kappa^{(q)}_{i[\mathbf{U}]}\,(\mathbf{k}_p(\mathbf{U})\,x^p\;+\;\boldsymbol{\theta}_0) \tag{3.1}$$

for some smooth functions $n^l_q(\mathbf{U})$. The functions $n^l_q(\mathbf{U})$ are then uniquely determined on Λ and we have $\mathrm{rk}\,||n^l_q(\mathbf{U})||=s$ by Definition 3.1.

Let us say that the full linearly independent subsystem of the regular left eigen-vectors of the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}$ with zero eigen-values can be given also by the variation derivatives of the functionals $I^1,\dots,I^{m+s}$ on Λ, which give exactly $m+s$ additional parameters $U^1,\dots,U^{m+s}$ to the parameters k^α_p. Relations (2.39) give the connection between the eigenvectors $\kappa^{(q)}_{i[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0)$ and $\zeta^{(\gamma)}_{i[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0)$, $\gamma=1,\dots,m+s$.

Easy to see then that for a complete Hamiltonian set of integrals $I^1,\dots,I^N$ we have also the relations

$$\mathrm{rk}\;||\omega^{\alpha\gamma}(\mathbf{U})||\;=\;m\;,\qquad \alpha=1,\dots,m\;,\quad \gamma=1,\dots,m+s \tag{3.2}$$

for the frequencies corresponding to the functionals $I^1,\dots,I^{m+s}$.

The construction which we are going to consider is in fact closely related with the construction of the Dirac restriction of a Poisson bracket on a submanifold. Let us describe here briefly the Dirac procedure. Using the terminology of finite-dimensional spaces we can say that the Dirac restriction of a Poisson bracket on a submanifold $\mathcal{N}^k\subset\mathcal{M}^n$ is associated with a special choice of coordinates in the neighborhood of the submanifold $\mathcal{N}^k$. The coordinates in the neighborhood of the submanifold $\mathcal{N}^k$ are divided into the "coordinates on the submanifold" $(U^1,\dots,U^k)$ and the constraints $(g^1,\dots,g^{n-k})$ which define the submanifold $\mathcal{N}^k$. It is assumed that the submanifold $\mathcal{N}^k$ is defined by the conditions

$$g^i(\mathbf{x})\;=\;0\quad,\quad i=1,\dots,n-k$$

while the functions $U^1(\mathbf{x}),\;\dots,\;U^k(\mathbf{x})$ on $\mathcal{M}^n$ play the role of a coordinate system on $\mathcal{N}^k$ after the restriction to this submanifold.

If the Hamiltonian flows generated by the functions $U^j(\mathbf{x})$ leave the submanifold $\mathcal{N}^k$ invariant, i.e. we have

$$\{U^j(\mathbf{x}),g^i(\mathbf{x})\}\;=\;0\quad\text{for}\quad\mathbf{g}(\mathbf{x})\;=\;0$$

then the pairwise Poisson brackets of the functions $U^j(\mathbf{x})$ define a Poisson tensor in the coordinates $(U^1,\dots,U^k)$ after restriction on $\mathcal{N}^k$ which is called the Dirac restriction of the Poisson bracket $\{\dots,\dots\}$ on the submanifold $\mathcal{N}^k\subset\mathcal{M}^n$.

In general, according to the procedure of Dirac, if we have some constraints $g^i(\mathbf{x})$ which define the submanifold $\mathcal{N}^k$ and some functions $U^j(\mathbf{x})$ which give a coordinate system on $\mathcal{N}^k$ we must find k linear combinations $\beta^j_s(\mathbf{U})\, g^s(\mathbf{x})$ at every point of $\mathcal{N}^k$, such that for the functions

$$\tilde{U}^j(\mathbf{x}) \;=\; U^j(\mathbf{x}) \,+\, \beta^j_s(\mathbf{U})\, g^s(\mathbf{x}) \quad , \quad j = 1, \dots, k$$

we have the relations

$$\{\tilde{U}^j(\mathbf{x}), g^i(\mathbf{x})\} \;=\; 0$$

for $\mathbf{g}(\mathbf{x}) \,=\, 0$.

The functions $\tilde{U}^j(\mathbf{x})$ take the same values that $U^j(\mathbf{x})$ at the points of $\mathcal{N}^k$ and we can define the Dirac bracket $\{\dots,\dots\}_D$ on $\mathcal{N}^k$ by the formula

$$\{U^i, U^j\}_D \;=\; \{\tilde{U}^i(\mathbf{x}), \tilde{U}^j(\mathbf{x})\}|_{\mathcal{N}^k}(\mathbf{U})$$

The functions $\beta^j_s(\mathbf{U})$ are determined from the linear systems

$$\{g^i(\mathbf{x}), g^s(\mathbf{x})\}|_{\mathcal{N}^k}\, \beta^j_s(\mathbf{U}) \;+\; \{g^i(\mathbf{x}), U^j(\mathbf{x})\}|_{\mathcal{N}^k} \;=\; 0 \quad , \quad i = 1, \dots, n-k$$

and we can also write

$$\{U^i, U^j\}_D \;=\; \{U^i(\mathbf{x}), U^j(\mathbf{x})\}|_{\mathcal{N}^k} \;-\; \beta^i_s(\mathbf{U})\, \{g^s(\mathbf{x}), g^q(\mathbf{x})\}|_{\mathcal{N}^k}\, \beta^j_q(\mathbf{U})$$

for the Dirac bracket on $\mathcal{N}^k$.

Let us now describe the procedure, which will be considered in our situation. We consider now system (2.5) which is Hamiltonian with respect to some local bracket (2.32) with a local Hamiltonian function of the form (2.33). We first introduce the extended space of fields

$$\boldsymbol{\varphi}(\mathbf{x}) \;\to\; \boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{x})$$

where the functions $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{x})$ are 2π-periodic with respect to each θ^α, and define the extended Poisson bracket:

$$\{\varphi^i(\boldsymbol{\theta}, \mathbf{x})\,,\, \varphi^j(\boldsymbol{\theta}', \mathbf{y})\} \;=\; \sum_{l_1,\dots,l_d} B^{ij}_{(l_1,\dots,l_d)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots)\; \delta^{(l_1)}(x^{l_1} - y^{l_1})\; \dots \delta^{(l_d)}(x^{l_d} - y^{l_d})\; \delta(\boldsymbol{\theta} - \boldsymbol{\theta}') \tag{3.3}$$

Let us then make the replacement $\mathbf{x} \to \mathbf{X} = \epsilon\mathbf{x}$ and introduce the Poisson bracket

$$\{\varphi^i(\boldsymbol{\theta}, \mathbf{X})\,,\, \varphi^j(\boldsymbol{\theta}', \mathbf{Y})\} \;=\; \sum_{l_1,\dots,l_d} \epsilon^{l_1+\dots+l_d}\, B^{ij}_{(l_1,\dots,l_d)}(\boldsymbol{\varphi},\, \epsilon\, \boldsymbol{\varphi}_{\mathbf{X}}, \dots)\; \delta^{(l_1)}(X^1 - Y^1) \dots \delta^{(l_d)}(X^d - Y^d)\, \delta(\boldsymbol{\theta} - \boldsymbol{\theta}') \tag{3.4}$$

on the space of fields $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X})$.

Let us describe now the submanifold $\mathcal{K}$ in the space of the functions $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X})$ which we are going to consider.

First, we will assume that the functions $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}) \in \mathcal{K}$ represent functions from the family Λ of the m-phase solutions of (2.5) with some parameters $\mathbf{U}(\mathbf{X})$ at every $\mathbf{X}$.

Second, we impose the relations

$$\frac{\partial k_p^\alpha(\mathbf{U}(\mathbf{X}))}{\partial X^l} \;=\; \frac{\partial k_l^\alpha(\mathbf{U}(\mathbf{X}))}{\partial X^p} \quad , \quad \alpha = 1, \dots, m \;\; , \;\; p, l = 1, \dots, d \tag{3.5}$$

for the functions $\mathbf{U}(\mathbf{X})$ parametrizing the solutions $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}) \in \Lambda$ from the submanifold $\mathcal{K}$.

More precisely, let us choose for simplicity the boundary conditions in the form $\mathbf{U}(X^1, 0, \dots, 0) \rightarrow \mathbf{U}_0$, $X^1 \rightarrow -\infty$, such that $k_1^\alpha(\mathbf{U}_0) = 0$ [2] and define the functions $S^\alpha(\mathbf{X})$ on $\mathcal{K}$ by the formula:

$$S^\alpha(\mathbf{X}) \;=\; \int_{-\infty}^{X^1} k_1^\alpha(X^{1\prime}, 0, \dots, 0)\, dX^{1\prime} \;+ \tag{3.6}$$

$$+ \int_0^{X^2} k_2^\alpha(X^1, X^{2\prime}, 0, \dots, 0)\, dX^{2\prime} \;+\; \dots \;+\; \int_0^{X^d} k_d^\alpha(X^1, \dots, X^{d-1}, X^{d\prime})\, dX^{d\prime}$$

We define then the functions $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}) \in \mathcal{K}$ by the formula

$$\varphi^i(\boldsymbol{\theta}, \mathbf{X}) \;=\; \Phi^i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} \,+\, \boldsymbol{\theta},\, \mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}) \right) \tag{3.7}$$

where the functions $k_p^\alpha(\mathbf{X}) \;=\; S^\alpha_{X^p}(\mathbf{X})$, and the additional parameters $(U^1, \dots, U^{m+s})$, introduced in the previous chapter, play now the role of parameters on the family Λ.

The functions $\{\Omega^\alpha(\mathbf{X}), U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})\}$ play the role of a coordinate system on the submanifold $\mathcal{K}$.

We have now to introduce the analogous coordinates in the vicinity of the submanifold $\mathcal{K}$.

Let us introduce the functionals $J^\nu(\mathbf{X})$ on the functions $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X})$ by the formula

$$J^\nu(\mathbf{X}) \;=\; \int_0^{2\pi} \!\!\dots \int_0^{2\pi} P^\nu(\boldsymbol{\varphi},\, \epsilon \boldsymbol{\varphi}_{\mathbf{X}},\, \epsilon^2 \boldsymbol{\varphi}_{\mathbf{XX}}, \dots)\, \frac{d^m \theta}{(2\pi)^m} \quad , \quad \nu = 1, \dots, N \tag{3.8}$$

and consider their values on the functions of the family $\mathcal{K}$.

We can write on $\mathcal{K}$:

$$J^\nu(\mathbf{X}) \;=\; U^\nu(\mathbf{X}) + \sum_{l \geq 1} \epsilon^l \, J^\nu_{(l)}(\mathbf{X}) \quad , \quad \nu = 1, \dots, N \tag{3.9}$$

where $J^\nu_{(l)}(\mathbf{X})$ - are polynomials in the derivatives $\mathbf{U}_{\mathbf{X}}$, $\mathbf{U}_{\mathbf{XX}}$, $\dots$ with coefficients depending on $\mathbf{U}$ and have grading degree l in terms of total number of derivations with respect to $\mathbf{X}$.

The higher terms in (3.9) are not uniquely defined on $\mathcal{K}$ due to relations (3.5). Let us assume here that the terms $J^\nu_{(l)}(\mathbf{X})$ are chosen in some definite way in every order $l \geq 1$. The corresponding choice will affect the definition of the functionals $\mathbf{U}(\mathbf{X})$ in the vicinity of $\mathcal{K}$ in the higher orders in ϵ ($l \geq 1$). As we will see, this choice will not be important in our further considerations.

The transformation (3.9) can then be inverted as a formal series in ϵ, such that we can write

[2] The case when the values $k_1^\alpha = 0$ are absent on the space of parameters $\mathbf{U}$ requires just a simple modification of the definition of $\mathbf{S}(\mathbf{X})$ which we for simplicity do not consider here.

$$U^{\nu}(\mathbf{X}) \;=\; J^{\nu}(\mathbf{X}) + \sum_{l\geq 1} \epsilon^{l}\, U^{\nu}_{(l)}(\mathbf{X}) \quad , \quad \nu = 1, \dots, N \tag{3.10}$$

on the functions of the submanifold $\mathcal{K}$. In formula (3.10) the functions $U^{\nu}_{(l)}$ are functions of $\mathbf{J}$, $\mathbf{J}_{\mathbf{X}}$, $\mathbf{J}_{\mathbf{XX}}$, ..., polynomial in the derivatives $\mathbf{J}_{\mathbf{X}}$, $\mathbf{J}_{\mathbf{XX}}$, ..., and having degree l in terms of the number of derivations w.r.t. $\mathbf{X}$.

We can define now the functionals $\mathbf{U}(\mathbf{X})$ on the whole functional space using the definition of the functionals $\mathbf{J}(\mathbf{X})$ and relations (3.10).

Let us put now the same boundary conditions for the functionals $k^{\alpha}_{1}(\mathbf{U})$ as before on the whole functional space. We can then consider also the functionals $S^{\alpha}[\mathbf{U}](\mathbf{X})$, given by (3.6), as the functionals defined in the vicinity of the submanifold $\mathcal{K}$ using the corresponding definition of the functionals $\mathbf{U}(\mathbf{X})$. On the submanifold $\mathcal{K}$ we will naturally have the relations

$$S^{\alpha}_{X^{p}}[\mathbf{U}](\mathbf{X}) \;=\; k^{\alpha}_{p}(\mathbf{U}(\mathbf{X}))$$

However, outside the submanifold $\mathcal{K}$ these relations are in general not true.

Let us introduce also the constraints $g^{i}(\boldsymbol{\theta}, \mathbf{X})$ defining the submanifold $\mathcal{K}$ by the conditions $g^{i}(\boldsymbol{\theta}, \mathbf{X}) = 0$, and numbered by the values of $\boldsymbol{\theta}$ and $\mathbf{X}$:

$$g^{i}(\boldsymbol{\theta}, \mathbf{X}) \;=\; \varphi^{i}(\boldsymbol{\theta}, \mathbf{X}) \;-\; \Phi^{i}\left(\frac{\mathbf{S}[\mathbf{U}[\mathbf{J}]](\mathbf{X})}{\epsilon} + \boldsymbol{\theta},\; S^{\alpha}_{\mathbf{X}}[\mathbf{U}[\mathbf{J}]],\, U^{1}[\mathbf{J}](\mathbf{X}),\, \dots,\, U^{m+s}[\mathbf{J}](\mathbf{X})\right) \tag{3.11}$$

The constraints $g^{i}(\boldsymbol{\theta}, \mathbf{X})$ are functionals on the whole extended space of fields $\varphi^{i}(\boldsymbol{\theta}, \mathbf{X})$ by virtue of the corresponding definition of the functionals $J^{\nu}(\mathbf{X})$.

The constraints (3.11) are not independent since the following relations hold identically for the "gradients" $\delta g^{i}(\boldsymbol{\theta}, \mathbf{X}) / \delta \varphi^{j}(\boldsymbol{\theta}', \mathbf{Y})$ on the submanifold $\mathcal{K}$:

$$\int\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi} \left.\frac{\delta S^{\alpha}(\mathbf{Z})}{\delta \varphi^{i}(\boldsymbol{\theta}, \mathbf{X})}\right|_{\mathcal{K}} \left.\frac{\delta g^{i}(\boldsymbol{\theta}, \mathbf{X})}{\delta \varphi^{j}(\boldsymbol{\theta}', \mathbf{Y})}\right|_{\mathcal{K}} \frac{d^{m}\theta}{(2\pi)^{m}}\, d^{d}X \;\equiv\; 0 \quad , \quad \alpha = 1, \dots, m \tag{3.12}$$

$$\int\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi} \left.\frac{\delta U^{\gamma}(\mathbf{Z})}{\delta \varphi^{i}(\boldsymbol{\theta}, \mathbf{X})}\right|_{\mathcal{K}} \left.\frac{\delta g^{i}(\boldsymbol{\theta}, \mathbf{X})}{\delta \varphi^{j}(\boldsymbol{\theta}', \mathbf{Y})}\right|_{\mathcal{K}} \frac{d^{m}\theta}{(2\pi)^{m}}\, d^{d}X \;\equiv\; 0 \quad , \quad \gamma = 1, \dots, m+s \tag{3.13}$$

Nevertheless, it will be convenient here not to choose an independent subsystem from (3.11) and keep the system of constraints in the form (3.11) keeping in mind the existence of identities (3.12) - (3.13).

Thus, we consider now the values of the functionals $[S^{\alpha}(\mathbf{X}), U^{1}(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}), g^{i}(\boldsymbol{\theta}, \mathbf{X})]$ as a coordinate system in the neighborhood of $\mathcal{K}$ with the relations (3.12) - (3.13). The values of the functionals $[S^{\alpha}(\mathbf{X}), U^{1}(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})]$ will be considered as a coordinate system on $\mathcal{K}$.

For the procedure of the averaging of bracket (2.32) we will need to find the pairwise brackets on $\mathcal{K}$ of the functionals, introduced above, according to the bracket (3.4).

The pairwise Poisson brackets of the functionals $J^{\nu}(\mathbf{X})$, $J^{\mu}(\mathbf{Y})$ have the form

$$\{J^{\nu}(\mathbf{X})\,,\,J^{\mu}(\mathbf{Y})\}\;=$$
$$=\sum_{l_1,\dots,l_d}\int_0^{2\pi}\!\!\dots\int_0^{2\pi} A^{\nu\mu}_{l_1\dots l_d}(\boldsymbol{\varphi}(\boldsymbol{\theta},\mathbf{X}),\epsilon\boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta},\mathbf{X}),\dots)\;\frac{d^m\theta}{(2\pi)^m}\;\epsilon^{l_1+\dots+l_d}\;\delta^{(l_1)}(X^1-Y^1)\;\dots\;\delta^{(l_d)}(X^d-Y^d)$$

where

$$A^{\nu\mu}_{0\dots0}(\boldsymbol{\varphi},\epsilon\boldsymbol{\varphi}_{\mathbf{X}},\dots)\;\equiv\;\epsilon\,\partial_{X^1}\,Q^{\nu\mu1}(\boldsymbol{\varphi},\epsilon\boldsymbol{\varphi}_{\mathbf{X}},\dots)\;+\;\dots\;+\;\epsilon\,\partial_{X^d}\,Q^{\nu\mu d}(\boldsymbol{\varphi},\epsilon\boldsymbol{\varphi}_{\mathbf{X}},\dots) \tag{3.14}$$

The Poisson brackets of the fields $\varphi^i(\boldsymbol{\theta},\mathbf{X})$ with the functionals $J^{\mu}(\mathbf{Y})$ can be written as

$$\{\varphi^i(\boldsymbol{\theta},\mathbf{X}),J^{\mu}(\mathbf{Y})\}\;=\sum_{l_1,\dots,l_d}\epsilon^{l_1+\dots+l_d}\,C^{i\mu}_{(l_1\dots l_d)}\,(\boldsymbol{\varphi}(\boldsymbol{\theta},\mathbf{X}),\epsilon\,\boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta},\mathbf{X}),\dots)\;\delta^{(l_1)}(X^1-Y^1)\dots\,\delta^{(l_d)}(X^d-Y^d) \tag{3.15}$$

with some smooth functions $C^{i\mu}_{(l_1\dots l_d)}(\boldsymbol{\varphi},\epsilon\,\boldsymbol{\varphi}_{\mathbf{X}},\dots)$.

We also have in this case

$$C^{i\mu}_{(0\dots0)}\,(\boldsymbol{\varphi}(\boldsymbol{\theta},\mathbf{X}),\epsilon\,\boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta},\mathbf{X}),\dots)\;\equiv\;S^{i\mu}\,(\boldsymbol{\varphi}(\boldsymbol{\theta},\mathbf{X}),\epsilon\,\boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta},\mathbf{X}),\dots) \tag{3.16}$$

by virtue of (2.35).

For any function of slow variable $q(\mathbf{Y})$ we can write

$$\left\{\varphi^i(\boldsymbol{\theta},\mathbf{X}),\int q(\mathbf{Y})\,J^{\mu}(\mathbf{Y})\,d^dY\right\}\;=\;\sum_{l_1,\dots,l_d}\epsilon^{l_1+\dots+l_d}\,C^{i\mu}_{(l_1\dots l_d)}\,(\boldsymbol{\varphi}(\boldsymbol{\theta},\mathbf{X}),\epsilon\,\boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta},\mathbf{X}),\dots)\;q_{l_1X^1\dots l_dX^d} \tag{3.17}$$

The leading term in expression (3.17) on $\mathcal{K}$ has the form

$$\left\{\varphi^i(\boldsymbol{\theta},\mathbf{X}),\int q(\mathbf{Y})\,J^{\mu}(\mathbf{Y})\,d^dY\right\}\Bigg|_{\mathcal{K}[0]}\;=\;C^{i\mu}_{(0)}\left(\boldsymbol{\Phi}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{X}\right),\dots\right)\;q(\mathbf{X})\;=$$
$$=\;q(\mathbf{X})\;S^{i\mu}\left(\boldsymbol{\Phi}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{X}\right),\dots\right)$$

where $S^{i\mu}(\boldsymbol{\varphi},\boldsymbol{\varphi}_{\mathbf{x}},\dots)$ is the flow (2.35) generated by the functional I^{μ}. According to (2.36) we can write then:

$$\left\{\varphi^i(\boldsymbol{\theta},\mathbf{X}),\int q(\mathbf{Y})\,J^{\mu}(\mathbf{Y})\,d^dY\right\}\Bigg|_{\mathcal{K}}\;=\;\omega^{\alpha\mu}(\mathbf{X})\;\Phi^i_{\theta^\alpha}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\;q(\mathbf{X})\;+\;O(\epsilon) \tag{3.18}$$

Similarly, for any smooth function $Q(\boldsymbol{\theta},\mathbf{X})$, 2π-periodic in each θ^{α}, we can write on the basis of (3.15)

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} Q\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{X}\right)\ \left.\{\varphi^i(\boldsymbol{\theta},\mathbf{X}), J^\mu(\mathbf{Y})\}\right|_{\mathcal{K}}\ \frac{d^m\theta}{(2\pi)^m}\ = \tag{3.19}$$

$$=\ \omega^{\alpha\mu}(\mathbf{X})\int_0^{2\pi}\!\!\dots\int_0^{2\pi}\ Q(\boldsymbol{\theta},\mathbf{X})\ \Phi^i_{\theta^\alpha}(\boldsymbol{\theta},\mathbf{U}(\mathbf{X}))\ \frac{d^m\theta}{(2\pi)^m}\ \delta(\mathbf{X}-\mathbf{Y})\ +\ O(\epsilon)$$

In view of relations (3.9) - (3.10) for the functionals $J^\mu(\mathbf{Y})$, $U^\mu(\mathbf{Y})$ we can also write

$$\left.\left\{\varphi^i(\boldsymbol{\theta},\mathbf{X}), \int q(\mathbf{Y})\,U^\mu(\mathbf{Y})\,d^dY\right\}\right|_{\mathcal{K}}\ =\ \omega^{\alpha\mu}(\mathbf{X})\ \Phi^i_{\theta^\alpha}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\ q(\mathbf{X})\ +\ O(\epsilon) \tag{3.20}$$

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} Q\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{X}\right)\ \left.\{\varphi^i(\boldsymbol{\theta},\mathbf{X}), U^\mu(\mathbf{Y})\}\right|_{\mathcal{K}}\ \frac{d^m\theta}{(2\pi)^m}\ = \tag{3.21}$$

$$=\ \omega^{\alpha\mu}(\mathbf{X})\int_0^{2\pi}\!\!\dots\int_0^{2\pi}\ Q(\boldsymbol{\theta},\mathbf{X})\ \Phi^i_{\theta^\alpha}(\boldsymbol{\theta},\mathbf{U}(\mathbf{X}))\ \frac{d^m\theta}{(2\pi)^m}\ \delta(\mathbf{X}-\mathbf{Y})\ +\ O(\epsilon)$$

In this case, by virtue of (2.37) we have

$$\left.\left\{\varphi^i(\boldsymbol{\theta},\mathbf{X}), \int q(\mathbf{Y})\,k^\alpha(\mathbf{U}(\mathbf{Y}))\,d^dY\right\}\right|_{\mathcal{K}}\ =\ O(\epsilon) \tag{3.22}$$

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} Q\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{X}\right)\ \left.\{\varphi^i(\boldsymbol{\theta},\mathbf{X})\,,\,k^\alpha(\mathbf{U}(\mathbf{Y}))\}\right|_{\mathcal{K}}\ \frac{d^m\theta}{(2\pi)^m}\ =\ O(\epsilon) \tag{3.23}$$

for fixed values of coordinates $[\mathbf{S}(\mathbf{Z}), U^1(\mathbf{Z}), \dots, U^{m+s}(\mathbf{Z})]$.

Let us prove now some lemmas about structure of the Poisson brackets on the submanifold $\mathcal{K}$ which we will need in the further consideration.

Lemma 3.1.

Let the values U^ν of the functionals I^ν on a complete regular family Λ of m-phase solutions of (2.5) be functionally independent and give a complete set of parameters on Λ, excluding the initial phases. Then for the Poisson brackets of the functionals $k^\alpha_p(\mathbf{U}(\mathbf{X}))$ and $J^\mu(\mathbf{Y})$ on $\mathcal{K}$ we have the following relations:

$$\{k^\alpha_p(\mathbf{U}(\mathbf{X})), J^\mu(\mathbf{Y})\}|_{\mathcal{K}}\ =\ \epsilon\ [\omega^{\alpha\mu}(\mathbf{U}(\mathbf{X}))\ \delta(\mathbf{X}-\mathbf{Y})]_{X^p}\ +\ O(\epsilon^2) \tag{3.24}$$

Proof.

The conditions of Lemma 3.1 coincide with the conditions of Lemmas 2.1 and 2.2. Consider the Hamiltonian flow generated by the functional $\int q(\mathbf{Y})\,J^\mu(\mathbf{Y})\,d^dY$ according to bracket (3.4) with a compactly supported function $q(\mathbf{Y})$ of the slow variable $\mathbf{Y}=\epsilon\mathbf{y}$.

According to (3.17) we can write at the "points" of the submanifold $\mathcal{K}$:

$$\varphi^i_t\ =\ q(\mathbf{X})\ \omega^{\beta\mu}(\mathbf{U}(\mathbf{X}))\ \Phi^i_{\theta^\beta}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\ +\ \epsilon\,\eta^i_{[q]}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{X}\right)\ +\ O(\epsilon^2) \tag{3.25}$$

with some (2π-periodic in each θ^{α}) functions $\eta^{i}_{[q]}(\boldsymbol{\theta}, \mathbf{X})$.

Let us introduce the functionals $k^{\alpha}_{p}(\mathbf{J}(\mathbf{X})) = k^{\alpha}_{p}(J^{1}(\mathbf{X}), \dots, J^{N}(\mathbf{X}))$ using the same functions $k^{\alpha}_{p}(\mathbf{U})$. According to (3.9) - (3.10) we can write

$$\{k^{\alpha}_{p}(\mathbf{U}(\mathbf{X})), J^{\mu}(\mathbf{Y})\}|_{\mathcal{K}} \; = \; \{k^{\alpha}_{p}(\mathbf{J}(\mathbf{X})), J^{\mu}(\mathbf{Y})\}|_{\mathcal{K}} \; + \; O(\epsilon^{2})$$

Consider the evolution of the functionals $k^{\alpha}_{p}(\mathbf{J}(\mathbf{X}))$ according to the flow generated by $\int q(\mathbf{Y}) \, J^{\mu}(\mathbf{Y}) \, d^{d}Y$ on the submanifold $\mathcal{K}$. Using relations (3.25) we can write

$$k^{\alpha}_{pt}(\mathbf{J}(\mathbf{X})) \;\; = \;\; \frac{\partial k^{\alpha}_{p}}{\partial U^{\nu}} \, (\mathbf{J}(\mathbf{X})) \;\; J^{\nu}_{t}(\mathbf{X}) \;\; =$$

$$= \frac{\partial k^{\alpha}_{p}}{\partial U^{\nu}} \, (\mathbf{J}(\mathbf{X})) \;\; q(\mathbf{X}) \; \omega^{\beta\mu}(\mathbf{U}(\mathbf{X})) \; \times$$
$$\times \; \int_{0}^{2\pi} \!\!\dots \int_{0}^{2\pi} \sum_{l_1,\dots,l_d} \Pi^{\nu(l_1 \dots l_d)}_{i} \left(\boldsymbol{\Phi} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right), \, \epsilon \frac{\partial}{\partial \mathbf{X}} \boldsymbol{\Phi} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right), \dots \right) \times$$
$$\times \; \epsilon^{l_1 + \dots + l_d} \; \frac{\partial^{l_1}}{\partial X^{1\, l_1}} \, \cdots \, \frac{\partial^{l_d}}{\partial X^{d\, l_d}} \, \Phi^{i}_{\theta^{\beta}} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right) \; \frac{d^{m}\theta}{(2\pi)^{m}} \; +$$

$$+ \; \epsilon \; \frac{\partial k^{\alpha}_{p}}{\partial U^{\nu}} \, (\mathbf{U}(\mathbf{X})) \, \sum_{q=1}^{d} \, \left(q(\mathbf{X}) \, \omega^{\beta\mu}(\mathbf{U}(\mathbf{X})) \right)_{X^{q}} \;\; \times$$
$$\times \; \int_{0}^{2\pi} \!\!\dots \int_{0}^{2\pi} \sum_{l_1,\dots,l_d} l_{q} \, \Pi^{\nu(l_1 \dots l_d)}_{i} \left(\boldsymbol{\Phi} \left(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right), \, k^{\gamma_1}_{1} \, \boldsymbol{\Phi}_{\theta^{\gamma_1}} \left(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right), \dots, \, k^{\gamma_d}_{d} \, \boldsymbol{\Phi}_{\theta^{\gamma_d}} \left(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right), \dots \right) \times$$
$$\times \; k^{\alpha^{1}_{1}}_{1} \dots k^{\alpha^{1}_{l_1}}_{1} \; \dots \; k^{\alpha^{q}_{1}}_{q} \dots k^{\alpha^{q}_{l_q - 1}}_{q} \; \dots \; k^{\alpha^{d}_{1}}_{d} \dots k^{\alpha^{d}_{l_d}}_{d} \; \Phi^{i}_{\theta^{\beta} \, \theta^{\alpha^{1}_{1}} \dots \theta^{\alpha^{1}_{l_1}} \, \dots \, \theta^{\alpha^{q}_{1}} \dots \theta^{\alpha^{q}_{l_q - 1}} \, \dots \, \theta^{\alpha^{d}_{1}} \dots \theta^{\alpha^{d}_{l_d}}} \left(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right) \; \frac{d^{m}\theta}{(2\pi)^{m}} \; +$$

$$+ \; \epsilon \; \frac{\partial k^{\alpha}_{p}}{\partial U^{\nu}} \, (\mathbf{U}(\mathbf{X})) \;\; \times$$
$$\times \; \int_{0}^{2\pi} \!\!\dots \int_{0}^{2\pi} \sum_{l_1,\dots,l_d} \Pi^{\nu(l_1 \dots l_d)}_{i} \left(\boldsymbol{\Phi} \left(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right), \, k^{\gamma_1}_{1} \, \boldsymbol{\Phi}_{\theta^{\gamma_1}} \left(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right), \dots, \, k^{\gamma_d}_{d} \, \boldsymbol{\Phi}_{\theta^{\gamma_d}} \left(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right), \dots \right) \; \times$$
$$\times \; k^{\alpha^{1}_{1}}_{1} \dots k^{\alpha^{1}_{l_1}}_{1} \; \dots \; k^{\alpha^{d}_{1}}_{d} \dots k^{\alpha^{d}_{l_d}}_{d} \; \eta^{i}_{[q] \, \theta^{\alpha^{1}_{1}} \dots \theta^{\alpha^{1}_{l_1}} \, \dots \, \theta^{\alpha^{d}_{1}} \dots \theta^{\alpha^{d}_{l_d}}} \left(\boldsymbol{\theta}, \mathbf{X} \right) \; \frac{d^{m}\theta}{(2\pi)^{m}} \;\; + \;\; O(\epsilon^{2})$$

It's not difficult to see that the first part of the above expression contains the integrals over $\boldsymbol{\theta}$ of the expressions $P^{\nu}_{\theta^{\beta}}(\boldsymbol{\theta}, \mathbf{X})$ and is equal to zero. It is easy to see also after integration by parts that the third part of the above expression represents the value

$$\epsilon \; \int_{0}^{2\pi} \!\!\dots \int_{0}^{2\pi} \frac{\partial k^{\alpha}_{p}}{\partial U^{\nu}} \, (\mathbf{U}(\mathbf{X})) \;\; \zeta^{(\nu)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta}) \;\; \eta^{i}_{[q]}(\boldsymbol{\theta}, \mathbf{X}) \;\; \frac{d^{m}\theta}{(2\pi)^{m}}$$

and is equal to zero according to Lemma 2.1.

Thus, we can write in the main order the expressions for the evolution of the functionals $k^{\alpha}_{p}(\mathbf{J}(\mathbf{X}))$ on the submanifold $\mathcal{K}$ in the form

$$k^{\alpha}_{pt}(\mathbf{J}(\mathbf{X})) \;=\; \epsilon \, \frac{\partial k^{\alpha}_{p}}{\partial U^{\nu}} \, (\mathbf{U}(\mathbf{X})) \sum_{q=1}^{d} \left(q(\mathbf{X}) \, \omega^{\beta\mu}(\mathbf{U}(\mathbf{X})) \right)_{X^{q}} \int_{0}^{2\pi} \!\! \dots \int_{0}^{2\pi} \sum_{l_1, \dots, l_d} l_{q} \, \Pi^{\nu (l_1 \dots l_d)}_{i\,[0]} \;\times$$
$$\times \;\; k_{1}^{\alpha^{1}_{1}} \dots k_{1}^{\alpha^{1}_{l_1}} \; \dots \; k_{q}^{\alpha^{q}_{1}} \dots k_{q}^{\alpha^{q}_{l_{q}-1}} \; \dots \; k_{d}^{\alpha^{d}_{1}} \dots k_{d}^{\alpha^{d}_{l_d}} \;\; \Phi^{i}_{\theta^{\beta}\,\theta^{\alpha^{1}_{1}} \dots \theta^{\alpha^{1}_{l_1}} \dots \theta^{\alpha^{q}_{1}} \dots \theta^{\alpha^{q}_{l_{q}-1}} \dots \theta^{\alpha^{d}_{1}} \dots \theta^{\alpha^{d}_{l_d}}} \; (\boldsymbol{\theta}, \mathbf{U}(\mathbf{X})) \; \frac{d^{m}\theta}{(2\pi)^{m}}$$

Using relations (2.18) we get then

$$k^{\alpha}_{pt}(\mathbf{J}(\mathbf{X})) \;=\; \epsilon \, \left[q(\mathbf{X}) \, \omega^{\alpha\mu}(\mathbf{U}(\mathbf{X})) \right]_{X^{p}} \;+\; O(\epsilon^{2})$$

i.e.

$$\{ k^{\alpha}(\mathbf{J}(\mathbf{X})), J^{\mu}(\mathbf{Y}) \}|_{\mathcal{K}} \;=\; \epsilon \, \omega^{\alpha\mu}(\mathbf{U}(\mathbf{X})) \; \delta_{X^{p}}(\mathbf{X} - \mathbf{Y}) \;+\; \epsilon \, \omega^{\alpha\mu}_{X^{p}} \; \delta(\mathbf{X} - \mathbf{Y}) \;+\; O(\epsilon^{2})$$

which implies (3.24) by virtue of relations (3.9) - (3.10).

Lemma 3.1 is proved.

Easy to see that according to Lemma 3.1 and relations (3.9) - (3.10) we can write also

$$\{ k^{\alpha}_{p}(\mathbf{U}(\mathbf{X})) \, , \, U^{\mu}(\mathbf{Y}) \}|_{\mathcal{K}} \;=\; \epsilon \, \left[\omega^{\alpha\mu}(\mathbf{U}(\mathbf{X})) \; \delta(\mathbf{X} - \mathbf{Y}) \right]_{X^{p}} \;+\; O(\epsilon^{2}) \tag{3.26}$$

on the submanifold $\mathcal{K}$.

Let us formulate now some corollaries following from Lemma 3.1.

1) Under the conditions of Lemma 3.1, we have also

$$\{ k^{\alpha}_{p}(\mathbf{X}) \, , \, k^{\beta}_{l}(\mathbf{Y}) \}|_{\mathcal{K}} \;=\; O(\epsilon^{2}) \tag{3.27}$$

for the functionals $\mathbf{k}_{p}(\mathbf{U}(\mathbf{X}))$.

Indeed, by virtue of (2.37) we have

$$\{ k^{\alpha}_{p}(\mathbf{X}) \, , \, k^{\beta}_{l}(\mathbf{Y}) \}|_{\mathcal{K}} \;=\; \{ k^{\alpha}_{p}(\mathbf{X}), U^{\mu}(\mathbf{Y}) \}|_{\mathcal{K}} \, \frac{\partial k^{\beta}_{l}}{\partial U^{\mu}}(\mathbf{U}(\mathbf{Y})) \;=\; \epsilon \, \omega^{\alpha\mu}(\mathbf{X}) \, k^{\beta}_{l,U^{\mu}}(\mathbf{X}) \, \delta_{X^{p}}(\mathbf{X} - \mathbf{Y}) \;+$$
$$+ \, \epsilon \, \omega^{\alpha\mu}(\mathbf{X}) \, \left(k^{\beta}_{l,U^{\mu}}(\mathbf{X}) \right)_{X^{p}} \; \delta(\mathbf{X} - \mathbf{Y}) \;+\; \epsilon \, \left(\omega^{\alpha\mu}(\mathbf{X}) \right)_{X^{p}} \; k^{\beta}_{l,U^{\mu}}(\mathbf{X}) \, \delta(\mathbf{X} - \mathbf{Y}) \;+\; O(\epsilon^{2}) \;=\; O(\epsilon^{2})$$

2) Using definition (3.6) of the functionals $S^{\alpha}(\mathbf{X})$ we can write on $\mathcal{K}$ under the conditions of Lemma 3.1:

$$\{ S^{\alpha}(\mathbf{X}) \, , \, J^{\mu}(\mathbf{Y}) \}|_{\mathcal{K}} \;=\; \epsilon \; \omega^{\alpha\mu}(\mathbf{U}(\mathbf{X})) \;\; \delta(\mathbf{X} - \mathbf{Y}) \;\;+\;\; O(\epsilon^{2}) \tag{3.28}$$

$$\{ S^{\alpha}(\mathbf{X}) \, , \, U^{\mu}(\mathbf{Y}) \}|_{\mathcal{K}} \;=\; \epsilon \; \omega^{\alpha\mu}(\mathbf{U}(\mathbf{X})) \;\; \delta(\mathbf{X} - \mathbf{Y}) \;+\; O(\epsilon^{2}) \tag{3.29}$$

$$\{k_p^\alpha(\mathbf{X})\,,\,S^\beta(\mathbf{Y})\}|_{\mathcal{K}} \;=\; O(\epsilon^2) \quad , \quad \{S^\alpha(\mathbf{X})\,,\,S^\beta(\mathbf{Y})\}|_{\mathcal{K}} \;=\; O(\epsilon^2) \tag{3.30}$$

for the functionals $S^\alpha(\mathbf{X})$.

Let us note now that in the proof of Lemma 3.1, we have not used the fact that the functionals $J^\mu(\mathbf{Y})$ belong to our special set of functionals (3.8) and used only the fact that the flow generated by the functional I^μ leaves invariant the family Λ generating linear shifts of θ_0^α with constant frequencies $\omega^{\alpha\mu}(\mathbf{U})$. We can therefore formulate here the following lemma:

Lemma 3.1′.

Let the values U^ν of the functionals I^ν on a complete regular family Λ of m-phase solutions of system (2.5) be functionally independent and give a complete set of parameters on Λ, excluding the initial phases. Let the flow generated by the functional

$$\tilde{I} \;=\; \int \tilde{P}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots)\; d^d x \tag{3.31}$$

leave invariant the family Λ generating linear shifts of θ_0^α with constant frequencies $\tilde{\omega}^\alpha(\mathbf{U})$.

Consider the functionals

$$\tilde{J}(\mathbf{X}) \;=\; \int_0^{2\pi}\!\!\dots\int_0^{2\pi} \tilde{P}(\boldsymbol{\varphi},\, \epsilon\boldsymbol{\varphi}_{\mathbf{X}},\, \epsilon^2\boldsymbol{\varphi}_{\mathbf{XX}}, \dots)\; \frac{d^m\theta}{(2\pi)^m} \tag{3.32}$$

Then for the Poisson brackets of the functionals $k^\alpha(\mathbf{U}(\mathbf{X}))$ and $\tilde{J}(\mathbf{Y})$ on $\mathcal{K}$ we have the relation:

$$\{k_p^\alpha(\mathbf{U}(\mathbf{X})), \tilde{J}(\mathbf{Y})\}|_{\mathcal{K}} \;=\; \epsilon\; [\tilde{\omega}^\alpha(\mathbf{U}(\mathbf{X}))\; \delta(\mathbf{X}-\mathbf{Y})]_{X^p} \;+\; O(\epsilon^2)$$

Proof of Lemma 3.1′ completely repeats the proof of Lemma 3.1.

Lemma 3.2.

Let the values U^ν of the functionals I^ν on a complete regular family Λ of m-phase solutions of system (2.5) be functionally independent and give a complete set of parameters on Λ, excluding the initial phases. Then for the constraints $g^i(\boldsymbol{\theta}, \mathbf{X})$ imposed by (3.11) and smooth compactly supported function $q(\mathbf{X})$ as well as smooth 2π-periodic in each θ^α function $Q(\boldsymbol{\theta}, \mathbf{X})$ we have the following relations on the submanifold $\mathcal{K}$:

$$\left\{ g^i(\boldsymbol{\theta}, \mathbf{X}), \int q(\mathbf{Y})\, J^\mu(\mathbf{Y})\, d^d Y \right\}\Bigg|_{\mathcal{K}} \;=\; O(\epsilon) \tag{3.33}$$

$$\left[\int_0^{2\pi}\!\!\dots\int_0^{2\pi} Q\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X}\right)\; \{g^i(\boldsymbol{\theta}, \mathbf{X})\,,\, J^\mu(\mathbf{Y})\}\; \frac{d^m\theta}{(2\pi)^m} \right]\Bigg|_{\mathcal{K}} \;=\; O(\epsilon) \tag{3.34}$$

Proof.

Indeed, by (3.18), (3.19), and Lemma 3.1 we have

$$\left\{ g^i(\boldsymbol{\theta}, \mathbf{X}), \int q(\mathbf{Y})\, J^\mu(\mathbf{Y})\, d^d Y \right\}\Bigg|_{\mathcal{K}[0]} \;=\; \Phi^i_{\theta^\alpha}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X})\right)\; \omega^{\alpha\mu}(\mathbf{X})\; q(\mathbf{X}) \;-$$

$$- \ \Phi^i_{\theta^\alpha}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X})\right) \left\{ S^\alpha(\mathbf{X})\,, \, \int q(\mathbf{Y})\, J^\mu(\mathbf{Y})\, d^dY \right\}\Bigg|_{\mathcal{K}[0]} \ \equiv \ 0$$

$$\left[\int_0^{2\pi} \!\!\dots \int_0^{2\pi} Q\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X}\right) \, \{ g^i(\boldsymbol{\theta}, \mathbf{X}), J^\mu(\mathbf{Y}) \} \ \frac{d^m\theta}{(2\pi)^m} \right]\Bigg|_{\mathcal{K}[0]} \ =$$

$$= \ \omega^{\alpha\mu}(\mathbf{X}) \int_0^{2\pi} \!\!\dots \int_0^{2\pi} Q(\boldsymbol{\theta}, \mathbf{X}) \ \Phi^i_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X})) \ \frac{d^m\theta}{(2\pi)^m} \ \ \delta(\mathbf{X} - \mathbf{Y}) \ -$$

$$- \int_0^{2\pi} \!\!\dots \int_0^{2\pi} Q(\boldsymbol{\theta}, \mathbf{X}) \ \Phi^i_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X})) \ \frac{d^m\theta}{(2\pi)^m} \quad \{ S^\alpha(\mathbf{X})\,, \, J^\mu(\mathbf{Y}) \}|_{\mathcal{K}[0]} \ \equiv \ 0$$

Lemma 3.2 is proved.

Similarly to the previous case, we can formulate also the following lemma:

Lemma 3.2′.

Let the values U^ν of the functionals I^ν on a complete regular family Λ of m-phase solutions of system (2.5) be functionally independent and give a complete set of parameters on Λ, excluding the initial phases. Let the flow generated by the functional (3.31) leave invariant the family Λ generating linear shifts of θ^α_0 with constant frequencies $\tilde{\omega}^\alpha(\mathbf{U})$. Then for the constraints $g^i(\boldsymbol{\theta}, \mathbf{X})$ and the functionals $\tilde{J}(\mathbf{X})$ imposed by (3.32) we have the following relations on the submanifold $\mathcal{K}$:

$$\left\{ g^i(\boldsymbol{\theta}, \mathbf{X}), \int q(\mathbf{Y})\, \tilde{J}(\mathbf{Y})\, d^dY \right\}\Bigg|_{\mathcal{K}} \ = \ O(\epsilon) \tag{3.35}$$

$$\left[\int_0^{2\pi} \!\!\dots \int_0^{2\pi} Q\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X}\right) \left\{ g^i(\boldsymbol{\theta}, \mathbf{X})\,, \, \tilde{J}(\mathbf{Y}) \right\} \ \frac{d^m\theta}{(2\pi)^m} \right]\Bigg|_{\mathcal{K}} \ = \ O(\epsilon) \tag{3.36}$$

under the same conditions for the functions $q(\mathbf{X})$ and $Q(\boldsymbol{\theta}, \mathbf{X})$.

Using transformations (3.9) - (3.10) we can also write

$$\left\{ g^i(\boldsymbol{\theta}, \mathbf{X}), \int q(\mathbf{Y})\, U^\mu(\mathbf{Y})\, d^dY \right\}\Bigg|_{\mathcal{K}} \ = \ O(\epsilon) \tag{3.37}$$

$$\left[\int_0^{2\pi} \!\!\dots \int_0^{2\pi} Q\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X}\right) \, \{ g^i(\boldsymbol{\theta}, \mathbf{X})\,, \, U^\mu(\mathbf{Y}) \} \ \frac{d^m\theta}{(2\pi)^m} \right]\Bigg|_{\mathcal{K}} \ = \ O(\epsilon) \tag{3.38}$$

for the functionals $U^\mu(\mathbf{Y})$.

The pairwise Poisson brackets of the constraints $g^i(\boldsymbol{\theta}, \mathbf{X})$, $g^j(\boldsymbol{\theta}', \mathbf{Y})$ on $\mathcal{K}$ can be written in the form:

$$\{ g^i(\boldsymbol{\theta}, \mathbf{X})\,, \, g^j(\boldsymbol{\theta}', \mathbf{Y}) \}|_{\mathcal{K}} \ = \tag{3.39}$$

$$= \ \{ \varphi^i(\boldsymbol{\theta}, \mathbf{X})\,, \, \varphi^j(\boldsymbol{\theta}', \mathbf{Y}) \}|_{\mathcal{K}} \ - \ \{ \varphi^i(\boldsymbol{\theta}, \mathbf{X})\,, \, U^\lambda(\mathbf{Y}) \}|_{\mathcal{K}} \ \Phi^j_{U^\lambda}\left(\frac{\mathbf{S}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{U}(\mathbf{Y})\right) \ -$$

$$- \, \Phi^i_{U^\nu}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X})\right) \ \{ U^\nu(\mathbf{X})\,, \, \varphi^j(\boldsymbol{\theta}', \mathbf{Y}) \}|_{\mathcal{K}} \ +$$

$$+\Phi^i_{U^\nu}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\{U^\nu(\mathbf{X})\,,\,U^\lambda(\mathbf{Y})\}|_{\mathcal{K}}\;\Phi^j_{U^\lambda}\left(\frac{\mathbf{S}(\mathbf{Y})}{\epsilon}+\boldsymbol{\theta}',\mathbf{U}(\mathbf{Y})\right)-$$

$$-\frac{1}{\epsilon}\{\varphi^i(\boldsymbol{\theta},\mathbf{X})\,,\,S^\beta(\mathbf{Y})\}|_{\mathcal{K}}\;\Phi^j_{\theta'^\beta}\left(\frac{\mathbf{S}(\mathbf{Y})}{\epsilon}+\boldsymbol{\theta}',\mathbf{U}(\mathbf{Y})\right)-$$

$$-\frac{1}{\epsilon}\Phi^i_{\theta^\alpha}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\{S^\alpha(\mathbf{X})\,,\,\varphi^j(\boldsymbol{\theta}',\mathbf{Y})\}|_{\mathcal{K}}+$$

$$+\frac{1}{\epsilon}\Phi^i_{\theta^\alpha}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\{S^\alpha(\mathbf{X})\,,\,U^\lambda(\mathbf{Y})\}|_{\mathcal{K}}\;\Phi^j_{U^\lambda}\left(\frac{\mathbf{S}(\mathbf{Y})}{\epsilon}+\boldsymbol{\theta}',\mathbf{U}(\mathbf{Y})\right)+$$

$$+\frac{1}{\epsilon}\Phi^i_{U^\nu}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\{U^\nu(\mathbf{X})\,,\,S^\beta(\mathbf{Y})\}|_{\mathcal{K}}\;\Phi^j_{\theta'^\beta}\left(\frac{\mathbf{S}(\mathbf{Y})}{\epsilon}+\boldsymbol{\theta}',\mathbf{U}(\mathbf{Y})\right)+$$

$$+\frac{1}{\epsilon^2}\Phi^i_{\theta^\alpha}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\{S^\alpha(\mathbf{X})\,,\,S^\beta(\mathbf{Y})\}|_{\mathcal{K}}\;\Phi^j_{\theta'^\beta}\left(\frac{\mathbf{S}(\mathbf{Y})}{\epsilon}+\boldsymbol{\theta}',\mathbf{U}(\mathbf{Y})\right)$$

Let us note that we assume that the parameters U^ν, U^λ represent here a full set of parameters $\mathbf{U}$ ($\nu,\lambda=1,\dots,N$) on Λ. The choice of the parameters U^ν is in fact not important due to the invariance of the corresponding expressions with respect to the substitutions $U^\nu = U^\nu(\tilde{\mathbf{U}})$ on Λ. So, for the sake of brevity, we put that the set $(U^1,\dots,U^N)$ represents the set of parameters $(k^\alpha_q,\,U^1,\dots,U^{m+s})$ or the set of averaged densities of the functionals I^ν introduced above.

For convenience let us introduce the functional

$$J_{[\mathbf{q}]} \;=\; \int J^\mu(\mathbf{Y})\;q_\mu(\mathbf{Y})\;d^dY$$

for any smooth compactly supported vector-function $\mathbf{q}(\mathbf{Y}) = (q_1(\mathbf{Y}),\dots,q_N(\mathbf{Y}))$. According to Lemma 3.2 we can write the relations $\{g^i(\boldsymbol{\theta},\mathbf{X})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}} = O(\epsilon)$ on the submanifold $\mathcal{K}$. More precisely, the first non-vanishing term of the Poisson bracket of $g^i(\boldsymbol{\theta},\mathbf{X})$ and $J_{[\mathbf{q}]}$ on $\mathcal{K}$ can be written as

$$\{g^i(\boldsymbol{\theta},\mathbf{X})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}[1]} \;=\; \{\varphi^i(\boldsymbol{\theta},\mathbf{X})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}[1]} \;-\; \Phi^i_{U^\nu}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\{U^\nu(\mathbf{X})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}[1]} \;-$$

$$-\;\Phi^i_{\theta^\alpha}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\{S^\alpha(\mathbf{X})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}[2]} \qquad (3.40)$$

Let us remind here that the indexes [1], [2] mean as before the terms of the corresponding graded expansion on $\mathcal{K}$ having degree 1 and 2 respectively.

As we mentioned already, the procedure of the averaging of a Poisson bracket is connected to some extend with the Dirac procedure of the restriction of a Poisson bracket onto a submanifold. The more detailed consideration of the connection of the averaging procedure with the Dirac procedure can be found in [25]. For the justification of the averaging procedure considered here we will need to investigate the solubility of the system

$$\hat{B}^{ij}_{[0]}(\mathbf{X})\,B_{j[\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X}) \;+\; A^i_{[1][\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X}) \;=\; 0 \qquad (3.41)$$

where

$$\hat{B}^{ij}_{[0]}(\mathbf{X}) \; = \; \sum_{l_1,\dots,l_d} B^{ij}_{(l_1\dots l_d)} \left(\mathbf{\Phi}(\boldsymbol{\theta},\mathbf{U}(\mathbf{X})),\, k_1^{\gamma_1}(\mathbf{X})\,\mathbf{\Phi}_{\theta^{\gamma_1}}(\boldsymbol{\theta},\mathbf{U}(\mathbf{X})),\dots,\, k_d^{\gamma_d}(\mathbf{X})\,\mathbf{\Phi}_{\theta^{\gamma_d}}(\boldsymbol{\theta},\mathbf{U}(\mathbf{X})),\,\dots\right) \times$$
$$\times \; k_1^{\alpha_1^1}(\mathbf{X})\dots k_1^{\alpha^1_{l_1}}(\mathbf{X})\,\dots\, k_d^{\alpha_1^d}(\mathbf{X})\dots k_d^{\alpha^d_{l_d}}(\mathbf{X}) \;\; \frac{\partial}{\partial\theta^{\alpha_1^1}}\dots\frac{\partial}{\partial\theta^{\alpha^1_{l_1}}}\,\dots\,\frac{\partial}{\partial\theta^{\alpha_1^d}}\dots\frac{\partial}{\partial\theta^{\alpha^d_{l_d}}} \qquad (3.42)$$

is the Hamiltonian operator (2.32) on the family of m-phase solutions of system (2.5) and

$$A^i_{[1][\mathbf{q}]}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{X}\right) \; = \; \{\varphi^i(\boldsymbol{\theta},\mathbf{X})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}[1]} \; - \; \Phi^i_{U^\nu}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon}+\boldsymbol{\theta},\mathbf{U}(\mathbf{X})\right)\{U^\nu(\mathbf{X})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}[1]} \qquad (3.43)$$

is a discrepancy connected with the first non-vanishing term of the Poisson bracket of $g^i(\boldsymbol{\theta},\mathbf{X})$ and $J_{[\mathbf{q}]}$ on $\mathcal{K}$. Let us note also that for the sake of brevity we denote again by U^ν the full set of parameters $\mathbf{U}$ on Λ as in the expression (3.39).

System (3.41) represents at every $\mathbf{X}$ a linear system of partial differential equations in $\boldsymbol{\theta}$ with periodic coefficients. For us the solubility of system (3.41) on the space of 2π-periodic in all θ^α functions will be important. Let us discuss now the properties of system (3.41).

It is easy to see that the operator $\hat{B}^{ij}_{[0]}(\mathbf{X}) \; = \; \hat{B}^{ij}_{[0]}(\mathbf{U}(\mathbf{X}))$ is a differential operator on the torus. It can be seen also that for special values of $\mathbf{k}_p(\mathbf{X})$ the foliation defined by the set of the vector fields $(\mathbf{k}_1(\mathbf{X}),\dots,\mathbf{k}_d(\mathbf{X}))$ can define the tori of the lower dimensions $\mathbb{T}^k \subset \mathbb{T}^m$ and even one-dimensional tori embedded in $\mathbb{T}^m$.

The operator $\hat{B}^{ij}_{[0]}(\mathbf{U})$ has in general finite number of "regular" eigenvectors with zero eigenvalues defined for all values of the parameters $\mathbf{U}$ and smoothly depending on the parameters. However, for special values of $\mathbf{U}$ the set of eigenvectors with zero eigenvalues can be infinite and is determined, in particular, by the dimension of closures of the foliation leaves in $\mathbb{T}^m$, defined by the vectors $\mathbf{k}_p(\mathbf{U})$.[3]

Let us define in the space of the parameters $\mathbf{U}$ the set $\mathcal{M}$, such that for all $\mathbf{U}\in\mathcal{M}$ the dimensions of the closures of the foliation leaves, defined by the set $\{\mathbf{k}_p(\mathbf{U})\}$ in $\mathbb{T}^m$ is equal to m. From the condition

$$\mathrm{rk}\;||\partial k^\alpha_p/\partial U^\nu|| \; = \; md$$

it follows that the set $\mathcal{M}$ is everywhere dense in the parameter space $\mathbf{U}$ and, moreover, has the full measure. We note also that the set $\mathbf{U}$ represents here the full set of parameters $\mathbf{U}=(U^1,\dots,U^N)$ on Λ excluding the initial phase shifts $\boldsymbol{\theta}_0$.

[3]For some special brackets (2.32) the differential part can be absent in the operator $\hat{B}^{ij}_{[0]}(\mathbf{U})$. The operator $\hat{B}^{ij}_{[0]}(\mathbf{U})$ reduces then to an ultralocal operator acting independently at every point of $\mathbb{T}^m$. As a rule, the matrix $B^{ij}_{[0]}(\mathbf{U})$ is non-degenerate in this case. For example, for the ultralocal Poisson bracket

$$\{\psi(\mathbf{x})\,,\,\bar{\psi}(\mathbf{y})\} \; = \; i\,\delta(\mathbf{x}-\mathbf{y})$$

for the multi-dimensional NLS equation

$$i\,\psi_t \; = \; \Delta\,\psi \, + \, \nu\,|\psi|^2\psi$$

we have exactly this situation. Easy to see that system (3.41) represents a simple algebraic system in this case and is trivially solvable. The multiphase situation is not different here from the single-phase case. However, for arbitrary brackets (2.32) the operators $\hat{B}^{ij}_{[0]}(\mathbf{U})$ have more general form described above.

In the study of the solubility of (3.41) we must first require the orthogonality of the functions $A^{i}_{[1][\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})$ to the "regular" eigenvectors of $\hat{B}^{ij}_{[0]}(\mathbf{U}(\mathbf{X}))$ with zero eigenvalues. Let us prove here the following lemma.

Lemma 3.3.

Let the values U^{ν} of the functionals I^{ν} on a complete regular family Λ of m-phase solutions of system (2.5) be functionally independent and give a complete set of parameters on Λ, excluding the initial phases. Then we have the relations

$$\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi} \kappa^{(q)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta})\; A^{i}_{[1][\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})\; \frac{d^{m}\theta}{(2\pi)^{m}} \;\equiv\; 0 \quad , \qquad q = 1,\dots,m+s \tag{3.44}$$

for the functions $A^{i}_{[1][\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})$ defined by (3.43).

Proof.

Let us first prove the following statement:

The values $\zeta^{(\gamma)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta})$, $\gamma = 1,\dots,m+s$, are orthogonal (for any $\mathbf{X}$) to the values $A^{i}_{[1][\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})$, i.e.

$$\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi} \zeta^{(\gamma)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta})\; A^{i}_{[1][\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})\; \frac{d^{m}\theta}{(2\pi)^{m}} \;\equiv\; 0 \quad , \qquad \gamma = 1,\dots,m+s \tag{3.45}$$

Indeed, according to (3.13) the convolution of the values $\delta U^{\gamma}(\mathbf{Z})/\delta\varphi^{i}(\boldsymbol{\theta},\mathbf{X})$ with the values $\{g^{i}(\boldsymbol{\theta},\mathbf{X})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}}$ identically vanish on $\mathcal{K}$. According to (3.9) - (3.10) the values $\delta U^{\gamma}(\mathbf{Z})/\delta\varphi^{i}(\boldsymbol{\theta},\mathbf{X})$ coincide in the leading order with the values $\delta J^{\gamma}(\mathbf{Z})/\delta\varphi^{i}(\boldsymbol{\theta},\mathbf{X})$ on $\mathcal{K}$.

Using the explicit expression for the quantities $\delta J^{\gamma}(\mathbf{Z})/\delta\varphi^{i}(\boldsymbol{\theta},\mathbf{X})$ on $\mathcal{K}$:

$$\left.\frac{\delta J^{\nu}(\mathbf{Z})}{\delta\varphi^{i}(\boldsymbol{\theta},\mathbf{X})}\right|_{\mathcal{K}} = \sum_{l_1,\dots,l_d} \Pi^{\gamma(l_1\dots l_d)}_{i}\left(\boldsymbol{\Phi}\left(\frac{\mathbf{S}(\mathbf{Z})}{\epsilon}+\boldsymbol{\theta},\mathbf{Z}\right),\dots\right)\,\epsilon^{l_1+\dots+l_d}\,\delta^{(l_1)}(Z^{1}-X^{1})\,\dots\,\delta^{(l_d)}(Z^{d}-X^{d})$$

we can write the corresponding convolution in the form of action of the operator

$$\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi} \frac{d^{m}\theta}{(2\pi)^{m}} \sum_{l_1,\dots,l_d} \Pi^{\gamma(l_1\dots l_d)}_{i}\left(\boldsymbol{\Phi}\left(\frac{\mathbf{S}(\mathbf{Z})}{\epsilon}+\boldsymbol{\theta},\mathbf{Z}\right),\dots\right)\,\epsilon^{l_1+\dots+l_d}\,\frac{d^{l_1}}{dZ^{1\,l_1}}\,\dots\,\frac{d^{l_d}}{dZ^{d\,l_d}} \tag{3.46}$$

on the distributions $\{g^{i}(\boldsymbol{\theta},\mathbf{Z})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}}$, which is given in the main order by the action of the operator

$$\begin{aligned}&\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi} \frac{d^{m}\theta}{(2\pi)^{m}} \sum_{l_1,\dots,l_d} \Pi^{\gamma(l_1\dots l_d)}_{i}\left(\boldsymbol{\Phi}\left(\frac{\mathbf{S}(\mathbf{Z})}{\epsilon}+\boldsymbol{\theta},\mathbf{Z}\right),\dots\right)\;\times\\ &\qquad\times\; k^{\alpha^{1}_{1}}_{1}(\mathbf{Z})\dots k^{\alpha^{1}_{l_1}}_{1}(\mathbf{Z})\,\dots\,k^{\alpha^{d}_{1}}_{d}(\mathbf{Z})\dots k^{\alpha^{d}_{l_d}}_{d}(\mathbf{Z})\;\; \frac{\partial}{\partial\theta^{\alpha^{1}_{1}}}\dots\frac{\partial}{\partial\theta^{\alpha^{1}_{l_1}}}\,\dots\,\frac{\partial}{\partial\theta^{\alpha^{d}_{1}}}\dots\frac{\partial}{\partial\theta^{\alpha^{d}_{l_d}}}\end{aligned} \tag{3.47}$$

The leading order of the brackets $\{g^{i}(\boldsymbol{\theta},\mathbf{Z})\,,\,J_{[\mathbf{q}]}\}$ on $\mathcal{K}$ is given by the value $\{g^{i}(\boldsymbol{\theta},\mathbf{Z})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}[1]}$, defined by formula (3.40). After integration by parts we get that the action of the operator (3.47) is given by the convolution w.r.t. $\boldsymbol{\theta}$ of the values $\{g^{i}(\boldsymbol{\theta},\mathbf{Z})\,,\,J_{[\mathbf{q}]}\}|_{\mathcal{K}[1]}$ with the values $\zeta^{(\gamma)}_{i[\mathbf{U}(\mathbf{Z})]}(\mathbf{S}(\mathbf{Z})/\epsilon+\boldsymbol{\theta})$.

We know also that the values $\zeta^{(\gamma)}_{i[\mathbf{U}(\mathbf{Z})]}(\mathbf{S}(\mathbf{Z})/\epsilon+\boldsymbol{\theta})$ are automatically orthogonal to the functions $\Phi^{i}_{\theta^\alpha}(\mathbf{S}(\mathbf{Z})/\epsilon+\boldsymbol{\theta},\mathbf{Z})$, so (after the replacement of $\mathbf{Z}$ to $\mathbf{X}$) we get relation (3.45).

Using now relations (2.39) we get the statement of the Lemma.

Lemma 3.3. is proved.

For a regular Hamiltonian family Λ and a complete Hamiltonian set of integrals $(I^1,\dots,I^N)$ we can also prove the following lemma:

Lemma 3.4.

Let the functions $B_{j[\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})$ satisfy conditions (3.41). Then the functions $B_{j[\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})$ automatically satisfy the conditions

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi}\Phi^{j}_{\theta^\alpha}(\boldsymbol{\theta},\,\mathbf{S}_{\mathbf{X}},\,U^1(\mathbf{X}),\dots,U^{m+s}(\mathbf{X}))\;B_{j[\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})\,\frac{d^m\theta}{(2\pi)^m}\;\equiv\;0\qquad,\qquad\alpha=1,\dots,m\tag{3.48}$$

Proof.

Indeed, the implementation of (3.41) implies the conditions

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi}\zeta^{(\gamma)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta})\;\hat{B}^{ij}_{[0]}(\mathbf{X})\,B_{j[\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})\,\frac{d^m\theta}{(2\pi)^m}\;=\;0\quad,\qquad\gamma=1,\dots,m+s$$

which is equivalent to

$$\omega^{\alpha\gamma}(\mathbf{U}(\mathbf{X}))\int_0^{2\pi}\!\!\dots\int_0^{2\pi}\Phi^{j}_{\theta^\alpha}(\boldsymbol{\theta},\mathbf{U}(\mathbf{X}))\;B_{j[\mathbf{q}]}(\boldsymbol{\theta},\mathbf{X})\,\frac{d^m\theta}{(2\pi)^m}\;=\;0$$

view the skew-symmetry of $\hat{B}^{ij}_{[0]}(\mathbf{X})$.

From the property (3.2) of the subset $(I^1,\dots,I^{m+s})$ of a complete Hamiltonian family of commuting functionals we immediately obtain now relations (3.48).

Lemma 3.4 is proved.

In the remaining part of the article the solubility of system (3.41) will play the basic role for the justification of the main results.

In what follows we consider separately the single-phase ($m=1$) and the multiphase ($m\geq 2$) cases.

The following lemma can be formulated for the single-phase case $m=1$:

Lemma 3.5.

Let Λ be a regular Hamiltonian family of single-phase solutions of (2.5) and $(I^1,\dots,I^N)$ be a complete Hamiltonian set of the first integrals of the form (2.8). Then the functions $B_{j[\mathbf{q}]}(\theta,\mathbf{X})$ can be found from system (3.41) and can be written in the form

$$B_{i[\mathbf{q}]}(\theta,\mathbf{X})\;=\;\beta^{\mu,p}_{i}(\theta,\mathbf{U}(\mathbf{X}))\;q_{\mu,X^p}(\mathbf{X})\;+\;\beta^{\mu,pq}_{i,\alpha}(\theta,\mathbf{U}(\mathbf{X}))\,S^{\alpha}_{X^pX^q}\;q_\mu(\mathbf{X})\;+\;\beta^{\mu,p}_{i,\gamma}(\theta,\mathbf{U}(\mathbf{X}))\,U^{\gamma}_{X^p}\;q_\mu(\mathbf{X})\tag{3.49}$$

(summation in $\mu=1,\dots,N$, $p,q=1,\dots,d$, $\gamma=1,\dots,m+s$) with smooth dependence on the parameters $\mathbf{U}(\mathbf{X})$.

Proof.

System (3.41) in the single-phase case is a system of ordinary differential equations in θ with a skew-symmetric operator $\hat{B}^{ij}_{[0]}(\mathbf{X})$. It is easy to see also that the right-hand side of system (3.41) has the form

$$-A^i_{[1][\mathbf{q}]}(\theta,\mathbf{X}) \;=\; \xi^{i\mu,p}(\theta,\mathbf{U}(\mathbf{X}))\, q_{\mu,X^p}(\mathbf{X}) + \xi^{i\mu,pq}_{\alpha}(\theta,\mathbf{U}(\mathbf{X}))\, S^{\alpha}_{X^pX^q}\; q_\mu(\mathbf{X}) + \xi^{i\mu,p}_{\gamma}(\theta,\mathbf{U}(\mathbf{X}))\, U^{\gamma}_{X^p}\; q_\mu(\mathbf{X})$$

with periodic in θ functions $\xi^{i\mu,p}(\theta,\mathbf{U}(\mathbf{X}))$, $\xi^{i\mu,pq}_{\alpha}(\theta,\mathbf{U}(\mathbf{X}))$, $\xi^{i\mu,p}_{\gamma}(\theta,\mathbf{U}(\mathbf{X}))$.

The orthogonality conditions (3.44) imply then the orthogonality of all the sets of functions $\xi^{i\mu,p}(\theta,\mathbf{U}(\mathbf{X}))$, $\xi^{i\mu,pq}_{\alpha}(\theta,\mathbf{U}(\mathbf{X}))$, $\xi^{i\mu,p}_{\gamma}(\theta,\mathbf{U}(\mathbf{X}))$ to the functions $\kappa^{(q)}_{i[\mathbf{U}(\mathbf{X})]}(\theta)$, such that system (3.41) can be split into independent inhomogeneous systems:

$$\hat{B}^{ij}_{[0]}(\mathbf{X})\;\beta^{\mu,p}_{j}(\theta,\mathbf{U}(\mathbf{X})) \;=\; \xi^{i\mu,p}(\theta,\mathbf{U}(\mathbf{X})) \tag{3.50}$$

$$\hat{B}^{ij}_{[0]}(\mathbf{X})\;\beta^{\mu,pq}_{j,\alpha}(\theta,\mathbf{U}(\mathbf{X})) \;=\; \xi^{i\mu,pq}_{\alpha}(\theta,\mathbf{U}(\mathbf{X})) \tag{3.51}$$

$$\hat{B}^{ij}_{[0]}(\mathbf{X})\;\beta^{\mu,p}_{j,\gamma}(\theta,\mathbf{U}(\mathbf{X})) \;=\; \xi^{i\mu,p}_{\gamma}(\theta,\mathbf{U}(\mathbf{X})) \tag{3.52}$$

defining functions (3.49).

All the systems (3.50) - (3.52) are systems of ordinary linear differential equations with periodic coefficients and a skew-symmetric operator $\hat{B}^{ij}_{[0]}(\mathbf{X})$. The zero modes of the operator $\hat{B}^{ij}_{[0]}(\mathbf{X})$ are given by the variation derivatives of annihilators of the bracket (2.32) and are orthogonal to the right-hand parts of (3.50) - (3.52) according to (3.1) and (3.44). Eigenfunctions of $\hat{B}^{ij}_{[0]}(\mathbf{X})$ form a basis in the space of 2π-periodic functions $\boldsymbol{\varphi}(\theta)$. Besides that, the nonzero eigenvalues of $\hat{B}^{ij}_{[0]}(\mathbf{X})$ are separated from zero in this case. Thus, the 2π-periodic functions $\beta^{\mu,p}_{i}(\theta,\mathbf{U}(\mathbf{X}))$, $\beta^{\mu,pq}_{i\,\alpha}(\theta,\mathbf{U}(\mathbf{X}))$, $\beta^{\mu,p}_{i\,\gamma}(\theta,\mathbf{U}(\mathbf{X}))$ can be found from systems (3.50) - (3.52) up to the variation derivatives of the annihilators of the bracket (2.32). If we impose additional conditions of orthogonality of $\beta^{\mu,p}_{i}(\theta,\mathbf{U}(\mathbf{X}))$, $\beta^{\mu,pq}_{i\,\alpha}(\theta,\mathbf{U}(\mathbf{X}))$, $\beta^{\mu,p}_{i\,\gamma}(\theta,\mathbf{U}(\mathbf{X}))$ to the variation derivatives of the annihilators of the bracket (2.32) on the manifold of single-phase solutions we can suggest a unique procedure of construction of these functions. The functions $\beta^{\mu,p}_{i}(\theta,\mathbf{U}(\mathbf{X}))$, $\beta^{\mu,pq}_{i\,\alpha}(\theta,\mathbf{U}(\mathbf{X}))$, $\beta^{\mu,p}_{i\,\gamma}(\theta,\mathbf{U}(\mathbf{X}))$ then depend smoothly on the parameters $\mathbf{U}(\mathbf{X})$, which implies the required properties for the functions $B_{j[\mathbf{q}]}(\theta,\mathbf{X})$.

Lemma 3.5 is proved.

Let us start now the investigation of system (3.41) in the general multi-phase case.

Lemma 3.6.

Let Λ be a regular Hamiltonian family of m-phase solutions of system (2.5) and $(I^1,\dots,I^N)$ be a complete Hamiltonian set of commuting first integrals of (2.5) having the form (2.8). Let for $\mathbf{U}\in\mathcal{M}$

$$\mathbf{v}^{(l)}_{[\mathbf{U}]}(\boldsymbol{\theta}) \;=\; \left(v^{(l)}_{1[\mathbf{U}]}(\boldsymbol{\theta}),\dots,v^{(l)}_{n[\mathbf{U}]}(\boldsymbol{\theta})\right) \quad , \qquad l=1,\dots,s$$

be the complete set of linearly independent eigenvectors of the operator $\hat{B}^{ij}_{[0]}(\mathbf{U})$ on the torus with zero eigenvalues, smoothly depending on $\boldsymbol{\theta}$. Then

1) The number of the vectors $\mathbf{v}^{(l)}_{[\mathbf{U}]}(\boldsymbol{\theta})$ is equal to the number of annihilators of the bracket (2.32) on the submanifold of m-phase solutions of (2.5);

2) The functions $A^i_{[1][\mathbf{q}]}(\boldsymbol{\theta}, \mathbf{X})$ are orthogonal to all the vectors $\mathbf{v}^{(l)}_{[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta})$, i.e.

$$\int_0^{2\pi} \!\!\dots \int_0^{2\pi} v^{(l)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta}) \,\, A^i_{[1][\mathbf{q}]}(\boldsymbol{\theta}, \mathbf{X}) \, \frac{d^m \theta}{(2\pi)^m} \,\,\equiv\,\, 0 \quad , \quad (\mathbf{U}(\mathbf{X}) \in \mathcal{M})$$

Proof.

Consider the values of $\mathbf{v}^{(l)}_{[\mathbf{U}]}(\boldsymbol{\theta})$ on any of the leaves of the foliation, defined by the set $\{\mathbf{k}_q(\mathbf{U})\}$ on the torus $\mathbb{T}^m$. According to the definition of regular Hamiltonian family of m-phase solutions of (2.5) the corresponding functions $v^{(l)}_{i[\mathbf{U}]}(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\theta}_0)$ should be the variation derivatives of some linear combination of annihilators of the bracket (2.32). We have then for a fixed value of $\boldsymbol{\theta}_0$:

$$v^{(l)}_{i[\mathbf{U}]}(\mathbf{k}_j \, x^j + \boldsymbol{\theta}_0) \;=\; \sum_{p=1}^{s} \alpha^l_p(\mathbf{U}, \boldsymbol{\theta}_0) \,\, \left. \frac{\delta N^p}{\delta \varphi^i(\mathbf{x})} \right|_{\boldsymbol{\varphi} = \boldsymbol{\Phi}(\mathbf{k}_j \, x^j + \boldsymbol{\theta}_0, \mathbf{U})}$$

From relation (3.1) we have then

$$v^{(l)}_{i[\mathbf{U}]}(\mathbf{k}_j(\mathbf{U}) \, x^j + \boldsymbol{\theta}_0) \;=\; \sum_{p=1}^{s} \sum_{q=1}^{m+s} \alpha^l_p(\mathbf{U}, \boldsymbol{\theta}_0) \,\, n^p_q(\mathbf{U}) \,\, \kappa^{(q)}_{i[\mathbf{U}]} \left(\mathbf{k}_j(\mathbf{U}) \, x^j \, + \, \boldsymbol{\theta}_0 \right) \tag{3.53}$$

By definition, for $\mathbf{U} \in \mathcal{M}$ the leaves of the foliation, defined by the set $\{\mathbf{k}_q(\mathbf{U})\}$, are everywhere dense in $\mathbb{T}^m$. Since both the left- and the right-hand parts of (3.53) are smooth functions on $\mathbb{T}^m$, we get then that they coincide on $\mathbb{T}^m$. We can put then $\alpha^l_p(\mathbf{U}, \boldsymbol{\theta}_0) = \alpha^l_p(\mathbf{U})$ and write

$$v^{(l)}_{i[\mathbf{U}]}(\boldsymbol{\theta}) \;\equiv\; \sum_{p=1}^{s} \sum_{q=1}^{m+s} \alpha^l_p(\mathbf{U}) \,\, n^p_q(\mathbf{U}) \,\, \kappa^{(q)}_{i[\mathbf{U}]} \, (\boldsymbol{\theta}) \tag{3.54}$$

It is easy to see also that any linear combination of the form (3.54) gives a regular eigenvector of the operator $\hat{B}^{ij}_{[0]}(\mathbf{U}(\mathbf{X}))$ with zero eigenvalue.

Statement (2) follows then from relation (3.44) in view of the representation (3.54).

Lemma 3.6 is proved.

We can see, in particular, that the values $A^i_{[1][\mathbf{q}]}(\boldsymbol{\theta}, \mathbf{X})$ are orthogonal at any $\mathbf{X}$ to all the "regular" eigen-vectors $v^{(l)}_{i[\mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta})$ of the operator $\hat{B}^{ij}_{[0]}(\mathbf{X})$ corresponding to the zero eigen-value view the regular dependence of the values $A^i_{[1][\mathbf{q}]}(\boldsymbol{\theta}, \mathbf{X})$ on the parameters $\mathbf{U}(\mathbf{X}) = (\mathbf{S}_{\mathbf{X}}, \, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}))$.

However, despite the presence of Lemma 3.6, study of system (3.41) is much more complicated in the multiphase case if compared with the single-phase case. Thus, the presence of "resonances" may lead to appearance of small eigenvalues of the operator $\hat{B}^{ij}_{[0]}(\mathbf{U}(\mathbf{X}))$ for some values of the parameters $\mathbf{U}(\mathbf{X})$. As a result, this circumstance may lead to insolubility of system (3.41) in the space of smooth periodic (in all θ^α) functions for the corresponding values of $\mathbf{U}(\mathbf{X})$. The solubility of system (3.41) is thus determined by the properties of the operator $\hat{B}^{ij}_{[0]}(\mathbf{U}(\mathbf{X}))$ for the given values of $\mathbf{U}(\mathbf{X})$.

The set of the "resonant" values of $\mathbf{U}$, as a rule, has measure zero in the full space of parameters. Let us prove here the following Theorem which shows that the procedure of the bracket averaging is in fact insensitive to the appearance of the resonant values of $\mathbf{U}$ and can be used in most multi-phase cases, as well as in the single-phase case.

Theorem 3.1.

Let system (2.5) be a local Hamiltonian system generated by the functional (2.33) in the local field-theoretic Hamiltonian structure (2.32). Let Λ be a regular Hamiltonian family of m-phase solutions of (2.5) and $(I^1, \dots, I^N)$ be a complete Hamiltonian set of commuting integrals (2.8) for this family.

Let the parameter space $\mathbf{U}$ of the family Λ have a dense set $\mathcal{S} \subset \mathcal{M}$ on which system (3.41) is solvable in the space of smooth 2π-periodic in each θ^α functions. Then the bracket

$$\big\{S^\alpha(\mathbf{X}) \, , \, S^\beta(\mathbf{Y})\big\} \; = \; 0 \quad , \qquad \{S^\alpha(\mathbf{X}) \, , \, U^\gamma(\mathbf{Y})\} \; = \; \omega^{\alpha\gamma} \left(\mathbf{S}_{\mathbf{X}}, \, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})\right) \; \delta(\mathbf{X} - \mathbf{Y}) \; ,$$

$$\{U^\gamma(\mathbf{X}) \, , \, U^\rho(\mathbf{Y})\} \; = \; \langle A^{\gamma\rho}_{10\dots0} \rangle \left(\mathbf{S}_{\mathbf{X}}, \, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})\right) \; \delta_{X^1}(\mathbf{X} - \mathbf{Y}) \; + \; \dots \; +$$

$$+ \; \langle A^{\gamma\rho}_{0\dots01} \rangle \left(\mathbf{S}_{\mathbf{X}}, \, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})\right) \; \delta_{X^d}(\mathbf{X} - \mathbf{Y}) \; +$$

$$+ \; \left[\langle Q^{\gamma\rho p} \rangle \left(\mathbf{S}_{\mathbf{X}}, \, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})\right) \right]_{X^p} \; \delta(\mathbf{X} - \mathbf{Y}) \quad , \qquad \gamma, \rho \, = \, 1, \dots, m+s \quad (3.55)$$

obtained with the aid of the functionals $(I^1, \dots, I^N)$, satisfies the Jacobi identity.

Proof.

As before, for a smooth compactly supported vector-valued function $\mathbf{q}(\mathbf{X}) = (q_1(\mathbf{X}), \dots, q_N(\mathbf{X}))$ we define the functional

$$J_{[\mathbf{q}]} \; = \; \int J^\nu(\mathbf{X}) \; q_\nu(\mathbf{X}) \; d^d X$$

(summation in $\nu = 1, \dots, N$).

Then, for arbitrary smooth, compactly supported in $\mathbf{X}$ and 2π-periodic in each θ^α functions $\tilde{\mathbf{Q}}(\boldsymbol{\theta}, \mathbf{X}) = (\tilde{Q}_1(\boldsymbol{\theta}, \mathbf{X}), \dots, \tilde{Q}_n(\boldsymbol{\theta}, \mathbf{X}))$ we define the functionals

$$Q_i(\boldsymbol{\theta}, \mathbf{X}) \; = \; \tilde{Q}_i(\boldsymbol{\theta}, \mathbf{X}) \; - \; \Phi^i_{\theta^\beta}(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X})) \; M^{\beta\gamma}(\mathbf{U}(\mathbf{X})) \int_0^{2\pi} \!\!\! \dots \int_0^{2\pi} \tilde{Q}_j(\boldsymbol{\theta}^\prime, \mathbf{X}) \, \Phi^j_{\theta^{\prime\gamma}}(\boldsymbol{\theta}^\prime, \mathbf{U}(\mathbf{X})) \; \frac{d^m \theta^\prime}{(2\pi)^m}$$

where the matrix $M^{\beta\gamma}(\mathbf{U})$ is the inverse of the matrix

$$\int_0^{2\pi} \!\!\! \dots \int_0^{2\pi} \sum_{i=1}^{n} \Phi^i_{\theta^\beta}(\boldsymbol{\theta}, \mathbf{U}) \; \Phi^i_{\theta^\gamma}(\boldsymbol{\theta}, \mathbf{U}) \; \frac{d^m \theta}{(2\pi)^m}$$

which is always defined according to the definition of a complete regular family of m-phase solutions of system (2.5).

By definition, the functions $Q_i(\boldsymbol{\theta}, \mathbf{X})$ are local functionals of $\mathbf{U}(\mathbf{X})$

$$Q_i(\boldsymbol{\theta}, \mathbf{X}) \; \equiv \; Q_i(\boldsymbol{\theta}, \mathbf{X}, \mathbf{U}(\mathbf{X}))$$

depending also on the arbitrary fixed functions $\tilde{\mathbf{Q}}(\boldsymbol{\theta}, \mathbf{X})$. Everywhere below we will assume that $\mathbf{Q}(\boldsymbol{\theta}, \mathbf{X})$ is a functional of this type defined with some function $\tilde{\mathbf{Q}}(\boldsymbol{\theta}, \mathbf{X})$.

Easy to see that the values of $Q_i(\boldsymbol{\theta}, \mathbf{X})$ with arbitrary $\tilde{\mathbf{Q}}(\boldsymbol{\theta}, \mathbf{X})$ represent for fixed values of the functionals $\mathbf{U}(\mathbf{Z})$ all possible smooth, compactly supported in $\mathbf{X}$ and 2π-periodic in each θ^α functions with the only restriction

$$\int_0^{2\pi} \dots \int_0^{2\pi} Q_i(\boldsymbol{\theta}, \mathbf{X}) \; \Phi^i_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{U}(\mathbf{X})) \; \frac{d^m\theta}{(2\pi)^m} \;\; = \;\; 0 \qquad , \;\; \forall \mathbf{X} \;\; , \qquad \alpha = 1, \dots, m \tag{3.56}$$

For the functionals $Q_i(\boldsymbol{\theta}, \mathbf{X})$ we define the functionals

$$g_{[\mathbf{Q}]} \;\; = \;\; \int \int_0^{2\pi} \dots \int_0^{2\pi} g^i(\boldsymbol{\theta}, \mathbf{X}) \; Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \; \frac{d^m\theta}{(2\pi)^m} \, d^d X$$

Now, for fixed functions $\mathbf{q}(\mathbf{X})$, $\mathbf{p}(\mathbf{X})$, and functional $\mathbf{Q}(\boldsymbol{\theta}, \mathbf{X})$ consider the Jacobi identity of the form

$$\left\{ g_{[\mathbf{Q}]} \, , \, \left\{ J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]} \right\} \right\} \; + \; \left\{ J_{[\mathbf{p}]} \, , \, \left\{ g_{[\mathbf{Q}]} \, , \, J_{[\mathbf{q}]} \right\} \right\} \; + \; \left\{ J_{[\mathbf{q}]} \, , \, \left\{ J_{[\mathbf{p}]} \, , \, g_{[\mathbf{Q}]} \right\} \right\} \;\; \equiv \;\; 0 \tag{3.57}$$

Expanding the values of the brackets $\{J^\nu(\mathbf{X}) \, , \, J^\mu(\mathbf{Y})\}$ in the neighborhood of the submanifold $\mathcal{K}$, we can write:

$$\{J^\nu(\mathbf{X}) \, , \, J^\mu(\mathbf{Y})\} \;\; = \;\; \{J^\nu(\mathbf{X}) \, , \, J^\mu(\mathbf{Y})\}|_{\mathcal{K}} \;\; +$$

$$+ \sum_{l_1, \dots, l_d} \left[\frac{\delta}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \int_0^{2\pi} \dots \int_0^{2\pi} A^{\nu\mu}_{l_1 \dots l_d} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}', \mathbf{X}), \epsilon \, \boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta}', \mathbf{X}), \dots \right) \; \frac{d^m\theta'}{(2\pi)^m} \right] \Bigg|_{\mathcal{K}} \times$$

$$\times \;\; g^k(\boldsymbol{\theta}, \mathbf{W}) \; \frac{d^m\theta}{(2\pi)^m} \, d^d W \;\; \epsilon^{l_1 + \dots + l_d} \;\; \delta^{(l_1)}(X^1 - Y^1) \; \dots \; \delta^{(l_d)}(X^d - Y^d) \;\; + \;\; O(\mathbf{g}^2)$$

where all the values on the submanifold $\mathcal{K}$ are calculated at the same values of the functionals $[\mathbf{U}(\mathbf{Z})]$ as for the original function $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X})$.

Let us introduce by definition

$$\frac{\delta \{ J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}} \;\; \equiv \tag{3.58}$$

$$\equiv \sum_{l_1, \dots, l_d} \int \left[\frac{\delta}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \int_0^{2\pi} \dots \int_0^{2\pi} \epsilon^{l_1 + \dots + l_d} \;\; A^{\nu\mu}_{l_1 \dots l_d} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}', \mathbf{X}), \epsilon \, \boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta}', \mathbf{X}), \dots \right) \; \frac{d^m\theta'}{(2\pi)^m} \right] \Bigg|_{\mathcal{K}} \times$$

$$\times \;\; q_\nu(\mathbf{X}) \, p_{\mu, \, l_1 X^1 \dots l_d X^d}(\mathbf{X}) \;\, d^d X$$

Note that the notations $\delta / \delta g^k(\boldsymbol{\theta}, \mathbf{W})$, generally speaking, are not natural in our situation because of the dependence of the chosen system of constraints. Nevertheless, the preservation of these notations can better clarify the algebraic structure of the further calculations.

The values defined by (3.58) can be represented in the form of the graded decompositions at $\epsilon \to 0$ on the submanifold $\mathcal{K}$. By virtue of (3.14) it is easy to conclude that the expansion in ϵ of the quantities (3.58) begins with the first degree in ϵ

$$\frac{\delta \{ J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}} \;\; = \;\; \epsilon \;\; \frac{\delta \{ J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}[1]} \;\; + \;\; \epsilon^2 \;\; \frac{\delta \{ J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}[2]} \;\; + \;\; \dots \tag{3.59}$$

The leading term of (3.59) can be divided into two parts, corresponding to the sets of the functions

$$\Big(\epsilon\, Q^{\nu\mu\, 1}_{X^1} (\boldsymbol{\varphi}, \epsilon\, \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \, , \, \dots \, , \, \epsilon\, Q^{\nu\mu\, d}_{X^d} (\boldsymbol{\varphi}, \epsilon\, \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \Big)$$

and

$$(\, \epsilon\, A^{\nu\mu}_{10\dots0} (\boldsymbol{\varphi}, \epsilon\, \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \, , \, \dots \, , \, \epsilon\, A^{\nu\mu}_{0\dots01} (\boldsymbol{\varphi}, \epsilon\, \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \,)$$

and containing the quantities $q_\nu(\mathbf{W}) p_\mu(\mathbf{W})$ and $q_\nu(\mathbf{W}) p_{\mu, W^l}(\mathbf{W})$ as local factors, respectively.

The values of $\{ J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]} \}$, are obviously invariant under transformations of the form

$$\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}) \quad \rightarrow \quad \boldsymbol{\varphi}(\boldsymbol{\theta} + \Delta \boldsymbol{\theta}, \mathbf{X}) \tag{3.60}$$

From the invariance of the functionals $\mathbf{U}(\mathbf{X})$ in such transformations, we can write for the corresponding increments of constraints (3.11)

$$\delta g^k (\boldsymbol{\theta}, \mathbf{X}) \;\; = \;\; \varphi^k (\boldsymbol{\theta} + \delta \boldsymbol{\theta}, \mathbf{X}) \; - \; \varphi^k (\boldsymbol{\theta}, \mathbf{X})$$

As a consequence, we can write on the submanifold $\mathcal{K}$

$$\int\!\!\int_0^{2\pi} \!\!\dots \int_0^{2\pi} \left. \frac{\delta \{ J_{[\mathbf{q}]}, J_{[\mathbf{p}]} \}}{\delta g^k (\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}} \; \Phi^k_{\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \, \mathbf{S}_{\mathbf{W}}, U^1(\mathbf{W}), \dots, U^{m+s}(\mathbf{W}) \right) \, \frac{d^m \theta}{(2\pi)^m} \, d^d W \;\; \equiv \;\; 0 \tag{3.61}$$

$\alpha = 1, \dots, m$.

The above relation is satisfied to all orders in ϵ. By the arbitrariness of the functions $q_\nu(\mathbf{X})$, relation (3.61) in the leading order can be strengthened. Namely, according to the remark about the form of the leading term of (3.59) we can write for any $\mathbf{W}$

$$\int_0^{2\pi} \!\!\dots \int_0^{2\pi} \left. \frac{\delta \{ J_{[\mathbf{q}]}, J_{[\mathbf{p}]} \}}{\delta g^k (\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \; \Phi^k_{\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \, \mathbf{S}_{\mathbf{W}}, U^1(\mathbf{W}), \dots, U^{m+s}(\mathbf{W}) \right) \, \frac{d^m \theta}{(2\pi)^m} \;\; \equiv \;\; 0 \tag{3.62}$$

$\alpha = 1, \dots, m$.

At the same time we have the relations

$$\left. \frac{\delta \{ J_{[\mathbf{q}]}, J_{[\mathbf{p}]} \}}{\delta S^\alpha (\mathbf{W})} \right|_{\mathcal{K}} \; = \; O(\epsilon) \quad\quad , \quad\quad \left. \frac{\delta \{ J_{[\mathbf{q}]}, J_{[\mathbf{p}]} \}}{\delta U^\gamma (\mathbf{W})} \right|_{\mathcal{K}} \; = \; O(\epsilon)$$

on the submanifold $\mathcal{K}$.

Quite similarly, we have the relation

$$\begin{aligned} \{ g_{[\mathbf{Q}]} \, , \, J_{[\mathbf{q}]} \} \;\; = & \int Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \; \{ g^i (\boldsymbol{\theta}, \mathbf{X}) \, , \, J_{[\mathbf{q}]} \} \; \frac{d^m \theta}{(2\pi)^m} \, d^d X \; + \\ & + \int g^i (\boldsymbol{\theta}, \mathbf{X}) \; Q_{i, \theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \; \frac{1}{\epsilon} \; \{ S^\alpha (\mathbf{X}) \, , \, J_{[\mathbf{q}]} \} \; \frac{d^m \theta}{(2\pi)^m} \, d^d X \; + \\ & + \int g^i (\boldsymbol{\theta}, \mathbf{X}) \; Q_{i, U^\nu} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \; \{ U^\nu (\mathbf{X}) \, , \, J_{[\mathbf{q}]} \} \; \frac{d^m \theta}{(2\pi)^m} \, d^d X \;\; = \end{aligned}$$

$$= \int Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \sum_{l_1,\dots,l_d} \epsilon^{l_1+\dots+l_d} \, C^{i\mu}_{(l_1\dots l_d)} \left(\boldsymbol{\varphi}(\boldsymbol{\theta},\mathbf{X}), \epsilon \, \boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta},\mathbf{X}), \dots \right) \frac{d^m\theta}{(2\pi)^m} \; q_{\mu,\, l_1 X^1 \dots l_d X^d}(\mathbf{X}) \, d^d X \; -$$

$$- \int Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \Phi^i_{k^\alpha_p} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \, \mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}) \right) \left\{ S^\alpha_{X^p}(\mathbf{X}) \, , \, J_{[\mathbf{q}]} \right\} \frac{d^m\theta}{(2\pi)^m} \; d^d X \; -$$

$$- \int Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \Phi^i_{U^\gamma} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \, \mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}) \right) \left\{ U^\gamma(\mathbf{X}) \, , \, J_{[\mathbf{q}]} \right\} \frac{d^m\theta}{(2\pi)^m} \; d^d X \; -$$

$$- \int Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \Phi^i_{\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \, \mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}) \right) \frac{1}{\epsilon} \left\{ S^\alpha(\mathbf{X}) \, , \, J_{[\mathbf{q}]} \right\} \frac{d^m\theta}{(2\pi)^m} \, d^d X \; +$$

$$+ \int g^i(\boldsymbol{\theta},\mathbf{X}) \; Q_{i,\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \frac{1}{\epsilon} \left\{ S^\alpha(\mathbf{X}) \, , \, J_{[\mathbf{q}]} \right\} \frac{d^m\theta}{(2\pi)^m} \; d^d X \; +$$

$$+ \int g^i(\boldsymbol{\theta},\mathbf{X}) \; Q_{i,U^\nu} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \left\{ U^\nu(\mathbf{X}) \, , \, J_{[\mathbf{q}]} \right\} \frac{d^m\theta}{(2\pi)^m} \, d^d X$$

Let us note that the summation in γ is made here for $\gamma = 1, \dots, m+s$, while the summation in ν is made for the full set of $U^\nu, \; \nu = 1, \dots, N$.

According to relations (3.56), we can actually see here that the fourth term in the above expression is identically equal to zero.

According to the form of the constraints, we have again in the neighborhood of $\mathcal{K}$:

$$\{ g_{[\mathbf{Q}]}, J_{[\mathbf{q}]} \} \;\; = \;\; \{ g_{[\mathbf{Q}]}, J_{[\mathbf{q}]} \} \big|_{\mathcal{K}} \; +$$

$$+ \left[\int Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}', \mathbf{X} \right) \; \times \right.$$
$$\left. \left. \times \sum_{l_1,\dots,l_d} \epsilon^{l_1+\dots+l_d} \; \frac{\delta C^{i\mu}_{(l_1\dots l_d)} (\boldsymbol{\varphi}(\boldsymbol{\theta}',\mathbf{X}), \epsilon \, \boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta}',\mathbf{X}), \dots)}{\delta \varphi^k(\boldsymbol{\theta},\mathbf{W})} \frac{d^m\theta'}{(2\pi)^m} \; q_{\mu, l_1 X^1 \dots l_d X^d}(\mathbf{X}) \; d^d X \right] \right|_{\mathcal{K}} \times$$
$$\times \; g^k(\boldsymbol{\theta},\mathbf{W}) \; \frac{d^m\theta}{(2\pi)^m} \; d^d W \; -$$

$$- \int Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}', \mathbf{X} \right) \Phi^i_{k^\alpha_p} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}', \, \mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}) \right) \frac{d^m\theta'}{(2\pi)^m} \; \times$$
$$\times \; \left. \frac{\left\{ S^\alpha_{X^p}(\mathbf{X}) \, , \, J_{[\mathbf{q}]} \right\}}{\delta \varphi^k(\boldsymbol{\theta},\mathbf{W})} \right|_{\mathcal{K}} d^d X \; \times \; g^k(\boldsymbol{\theta},\mathbf{W}) \; \frac{d^m\theta}{(2\pi)^m} \; d^d W \; -$$

$$- \int Q_i(\boldsymbol{\theta}',\mathbf{X}) \; \Phi^i_{U^\gamma} \left(\boldsymbol{\theta}', \, \mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}) \right) \; \frac{d^m\theta'}{(2\pi)^m} \; \left. \frac{\delta \{ U^\gamma(\mathbf{X}), J_{[\mathbf{q}]} \}}{\delta \varphi^k(\boldsymbol{\theta},\mathbf{W})} \right|_{\mathcal{K}} d^d X \; \times$$
$$\times \; g^k(\boldsymbol{\theta},\mathbf{W}) \; \frac{d^m\theta}{(2\pi)^m} \; d^d W \; +$$

$$+ \int Q_{k,\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \frac{1}{\epsilon} \; \left\{ S^\alpha(\mathbf{W}) \, , \, J_{[\mathbf{q}]} \right\} \Big|_{\mathcal{K}} \times \; g^k(\boldsymbol{\theta}, \mathbf{W}) \; \frac{d^m \theta}{(2\pi)^m} \, d^d W \; +$$

$$+ \int Q_{i,U^\nu} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \; \left\{ U^\nu(\mathbf{W}) \, , \, J_{[\mathbf{q}]} \right\} \Big|_{\mathcal{K}} \times \; g^k(\boldsymbol{\theta}, \mathbf{W}) \; \frac{d^m \theta}{(2\pi)^m} \, d^d W \quad + \quad O(\mathbf{g}^2)$$

provided that all the values on the submanifold $\mathcal{K}$ are calculated at the same values of the functionals $[\mathbf{U}(\mathbf{Z})]$.

For the quantities $\{ g_{[\mathbf{Q}]}, J_{[\mathbf{q}]} \}$ we can introduce by definition

$$\frac{\delta \{ g_{[\mathbf{Q}]}, J_{[\mathbf{q}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \bigg|_{\mathcal{K}} \quad \equiv \tag{3.63}$$

$$\equiv \; \left[\int Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}', \mathbf{X} \right) \; \times \right.$$

$$\left. \times \sum_{l_1, \dots, l_d} \epsilon^{l_1 + \dots + l_d} \; \frac{\delta C^{i\mu}_{(l_1 \dots l_d)} (\boldsymbol{\varphi}(\boldsymbol{\theta}', \mathbf{X}), \epsilon \, \boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta}', \mathbf{X}), \dots)}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \; q_{\mu, l_1 X^1 \dots l_d X^d}(\mathbf{X}) \right] \Bigg|_{\mathcal{K}} \frac{d^m \theta'}{(2\pi)^m} \; d^d X \; -$$

$$- \int Q_i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}', \mathbf{X} \right) \Phi^i_{k^\alpha_p} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}', \, \mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}) \right) \frac{d^m \theta'}{(2\pi)^m} \; \frac{\{ S^\alpha_{X^p}(\mathbf{X}), J_{[\mathbf{q}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \bigg|_{\mathcal{K}} d^d X \; -$$

$$- \int Q_i(\boldsymbol{\theta}', \mathbf{X}) \; \Phi^i_{U^\gamma} \left(\boldsymbol{\theta}', \, \mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}) \right) \; \frac{d^m \theta'}{(2\pi)^m} \quad \frac{\delta \{ U^\gamma(\mathbf{X}), J_{[\mathbf{q}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \bigg|_{\mathcal{K}} d^d X \; +$$

$$+ Q_{k,\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \; \frac{1}{\epsilon} \; \left\{ S^\alpha(\mathbf{W}) \, , \, J_{[\mathbf{q}]} \right\} \Big|_{\mathcal{K}} \; +$$

$$+ \; Q_{i,U^\nu} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \; \left\{ U^\nu(\mathbf{W}) \, , \, J_{[\mathbf{q}]} \right\} \Big|_{\mathcal{K}}$$

The quantities $\delta \{ g_{[\mathbf{Q}]}, J_{[\mathbf{q}]} \} / \delta g^k(\boldsymbol{\theta}, \mathbf{W}) |_{\mathcal{K}}$ have the order $O(1)$ at $\epsilon \to 0$. We have also

$$C^{i\mu}_{(0 \dots 0)} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}', \mathbf{X}), \epsilon \, \boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta}', \mathbf{X}), \dots \right) \; \equiv \; S^{i\mu} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}', \mathbf{X}), \epsilon \, \boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta}', \mathbf{X}), \dots \right)$$

according to relation (3.16).

Let us introduce the functions

$$S^{i\mu}_{k(l_1 \dots l_d)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots) \; \equiv \; \frac{\partial S^{i\mu}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots)}{\partial \varphi^k_{l_1 x^1 \dots l_d x^d}}$$

Using now relations (3.28) and (3.59) we can write for the leading term of (3.63)

$$\frac{\delta \{ g_{[\mathbf{Q}]}, J_{[\mathbf{q}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \bigg|_{\mathcal{K}[0]} \quad =$$

$$= \sum_{l_1, \dots, l_d} (-1)^{l_1 + \dots + l_d} \; k_1^{\alpha^1_1}(\mathbf{W}) \dots k_1^{\alpha^1_{l_1}}(\mathbf{W}) \; \dots \; k_d^{\alpha^d_1}(\mathbf{W}) \dots k_d^{\alpha^d_{l_d}}(\mathbf{W}) \; \frac{\partial}{\partial \theta^{\alpha^1_1}} \dots \frac{\partial}{\partial \theta^{\alpha^1_{l_1}}} \; \dots \; \frac{\partial}{\partial \theta^{\alpha^d_1}} \dots \frac{\partial}{\partial \theta^{\alpha^d_{l_d}}} \times$$

$$\times \; \left[Q_i \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) S^{i\mu}_{k(l_1 \dots l_d)} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{W}), \epsilon \boldsymbol{\varphi}_{\mathbf{W}}(\boldsymbol{\theta}, \mathbf{W}), \dots \right) \Big|_{\mathcal{K}[0]} \right] q_\mu(\mathbf{W}) \; +$$

$$+ \ \ \omega^{\alpha\mu}(\mathbf{W}) \ q_\mu(\mathbf{W}) \ Q_{k,\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \tag{3.64}$$

where

$$S^{i\mu}_{k(l_1 \dots l_d)} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{W}), \epsilon \boldsymbol{\varphi}_{\mathbf{W}}(\boldsymbol{\theta}, \mathbf{W}), \dots \right) \Big|_{\mathcal{K}[0]} \equiv S^{i\mu}_{k(l_1 \dots l_d)} \left(\mathbf{\Phi} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{S}_{\mathbf{W}}, U^1(\mathbf{W}), \dots, U^{m+s}(\mathbf{W}) \right), \dots \right)$$

Note one more property of the values $\delta\{g_{[\mathbf{Q}]}, J_{[\mathbf{q}]}\}/\delta g^k(\boldsymbol{\theta}, \mathbf{W})|_{\mathcal{K}[0]}$. As we saw earlier, the values $\{g_{[\mathbf{Q}]}, J_{[\mathbf{q}]}\}$ are of the order $O(\epsilon)$ at $\epsilon \to 0$ on the submanifold $\mathcal{K}$. This property is preserved also under the overall shift of the initial phase (3.60).

Indeed, for

$$\varphi^i(\boldsymbol{\theta}, \ \mathbf{X}) \ \ = \ \ \Phi^i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta} + \Delta \boldsymbol{\theta}, \ \mathbf{S}_{\mathbf{X}}, \ U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}) \right) \tag{3.65}$$

we can write

$$\left\{ g_{[\mathbf{Q}]}, J_{[\mathbf{q}]} \right\} \ = \ \int Q_{i,\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{Z})}{\epsilon} + \boldsymbol{\theta}, \mathbf{Z} \right) \ \times$$
$$\times \ \frac{1}{\epsilon} \left\{ S^\alpha(\mathbf{Z}), J_{[\mathbf{q}]} \right\} \ \Phi^i \left(\frac{\mathbf{S}(\mathbf{Z})}{\epsilon} + \boldsymbol{\theta} + \Delta\boldsymbol{\theta}, \ \mathbf{S}_{\mathbf{Z}}, \ U^1(\mathbf{Z}), \dots, U^{m+s}(\mathbf{Z}) \right) \ \frac{d^m \theta}{(2\pi)^m} \, d^d Z \ +$$

$$+ \int Q_i \left(\frac{\mathbf{S}(\mathbf{Z})}{\epsilon} + \boldsymbol{\theta}, \mathbf{Z} \right) C^{i\mu}_{(0 \dots 0)} \left(\mathbf{\Phi} \left(\frac{\mathbf{S}(\mathbf{Z})}{\epsilon} + \boldsymbol{\theta} + \Delta\boldsymbol{\theta}, \ \mathbf{S}_{\mathbf{Z}}, \ U^1(\mathbf{Z}), \dots, U^{m+s}(\mathbf{Z}) \right), \dots \right) \times$$
$$\times \ q_\mu(\mathbf{Z}) \, \frac{d^m \theta}{(2\pi)^m} \, d^d Z \ \ + \ \ O(\epsilon)$$

Because of the invariance under translations (3.60) the value of the bracket $\{S^\alpha(\mathbf{Z}), J_{[\mathbf{q}]}\}$ on the functions (3.65) is equal to its value on $\mathcal{K}$

$$\left\{ S^\alpha(\mathbf{Z}) \, , \ J_{[\mathbf{q}]} \right\} \ \ = \ \ \epsilon \, \omega^{\alpha\nu}(\mathbf{Z}) \, q_\nu(\mathbf{Z}) \ \ + \ \ O(\epsilon^2)$$

Similarly, we have on the functions (3.65)

$$C^{i\mu}_{(0 \dots 0)} \left(\mathbf{\Phi} \left(\frac{\mathbf{S}(\mathbf{Z})}{\epsilon} + \boldsymbol{\theta} + \Delta\boldsymbol{\theta}, \ \mathbf{S}_{\mathbf{Z}}, \ U^1(\mathbf{Z}), \dots, U^{m+s}(\mathbf{Z}) \right), \dots \right) \ =$$
$$= \ \omega^{\alpha\mu}(\mathbf{Z}) \ \Phi^i_{\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{Z})}{\epsilon} + \boldsymbol{\theta} + \Delta\boldsymbol{\theta}, \ \mathbf{S}_{\mathbf{Z}}, \ U^1(\mathbf{Z}), \dots, U^{m+s}(\mathbf{Z}) \right)$$

We then obtain

$$\left\{ g_{[\mathbf{Q}]}, J_{[\mathbf{q}]} \right\} \ = \ \int q_\mu(\mathbf{Z}) \ \omega^{\alpha\mu}(\mathbf{Z}) \ \frac{\partial}{\partial \theta^\alpha} \left[Q_i(\boldsymbol{\theta}, \mathbf{Z}) \, \Phi^i(\boldsymbol{\theta} + \Delta\boldsymbol{\theta}, \mathbf{Z}) \right] \ \frac{d^m \theta}{(2\pi)^m} \, d^d Z \ + \ O(\epsilon) \ = \ O(\epsilon)$$

As a consequence, we can write

$$\int\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \left. \frac{\delta\{g_{[\mathbf{Q}]}, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[0]} \Phi^k_{\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta},\, \mathbf{S}_{\mathbf{W}},\, U^1(\mathbf{W}), \dots, U^{m+s}(\mathbf{W}) \right) \frac{d^m\theta}{(2\pi)^m}\, d^dW \;\equiv\; 0 \tag{3.66}$$

$\alpha = 1, \dots, m$, for the main part of $\delta\{g_{[\mathbf{Q}]}, J_{[\mathbf{q}]}\}/\delta g^k(\boldsymbol{\theta}, \mathbf{W})$ on $\mathcal{K}$.

Using again the fact that relation (3.66) contains arbitrary functions $q_\mu(\mathbf{W})$, appearing in the integrand expression in the form of local factors, we can rewrite (3.66) in the stronger form

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \left. \frac{\delta\{g_{[\mathbf{Q}]}, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[0]} \Phi^k_{\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta},\, \mathbf{S}_{\mathbf{W}},\, U^1(\mathbf{W}), \dots, U^{m+s}(\mathbf{W}) \right) \frac{d^m\theta}{(2\pi)^m} \;\equiv\; 0 \quad , \ \forall\, \mathbf{W} \tag{3.67}$$

$\alpha = 1, \dots, m$.

At the same time we have

$$\left. \frac{\delta\{g_{[\mathbf{Q}]}, J_{[\mathbf{q}]}\}}{\delta S^\alpha(\mathbf{W})} \right|_{\mathcal{K}} \;=\; O(\epsilon) \quad , \quad \left. \frac{\delta\{g_{[\mathbf{Q}]}, J_{[\mathbf{q}]}\}}{\delta U^\gamma(\mathbf{W})} \right|_{\mathcal{K}} \;=\; O(\epsilon)$$

on the submanifold $\mathcal{K}$.

We now turn back to the Jacobi identity (3.57) for the functionals $g_{[\mathbf{Q}]}$, $J_{[\mathbf{q}]}$, and $J_{[\mathbf{p}]}$. Using expansions of the values $\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}$, $\{g_{[\mathbf{Q}]}, J_{[\mathbf{q}]}\}$, $\{g_{[\mathbf{Q}]}, J_{[\mathbf{p}]}\}$ in terms of $g^k(\boldsymbol{\theta}, \mathbf{W})$, $S^\alpha(\mathbf{W})$, $U^\gamma(\mathbf{W})$ near $\mathcal{K}$, it is not difficult to see that after the restriction on $\mathcal{K}$ the leading term ($\sim \epsilon$) of relation (3.57) can be written as

$$\int \left. \{g_{[\mathbf{Q}]}\,,\, g^k(\boldsymbol{\theta}, \mathbf{W})\} \right|_{\mathcal{K}[0]} \left. \frac{\delta\{J_{[\mathbf{q}]}\,,\, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \frac{d^m\theta}{(2\pi)^m}\, d^dW \;+$$

$$+ \int \left. \{J_{[\mathbf{p}]}\,,\, g^k(\boldsymbol{\theta}, \mathbf{W})\} \right|_{\mathcal{K}[1]} \left. \frac{\delta\{g_{[\mathbf{Q}]}\,,\, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[0]} \frac{d^m\theta}{(2\pi)^m}\, d^dW \;-$$

$$- \int \left. \{J_{[\mathbf{q}]}\,,\, g^k(\boldsymbol{\theta}, \mathbf{W})\} \right|_{\mathcal{K}[1]} \left. \frac{\delta\{g_{[\mathbf{Q}]}\,,\, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[0]} \frac{d^m\theta}{(2\pi)^m}\, d^dW \;\equiv\; 0$$

The above identity can again be written in a stronger form. Namely, making the change $\tilde{Q}_i(\boldsymbol{\theta}, \mathbf{W}) \rightarrow \tilde{Q}_i(\boldsymbol{\theta}, \mathbf{W})\, \mu_i(\mathbf{W})$, we get the corresponding change: $Q_i(\boldsymbol{\theta}, \mathbf{W}) \rightarrow Q_i(\boldsymbol{\theta}, \mathbf{W})\, \mu_i(\mathbf{W})$, where $\mu_i(\mathbf{W})$ are arbitrary smooth functions of $\mathbf{W}$. From the form of the pairwise brackets of constraints given by (3.39), and (3.64), it is easy to see then that the integrands are smooth functions of $\boldsymbol{\theta}$ and $\mathbf{W}$, containing $\mu_i(\mathbf{W})$ in the form of local factors. By the arbitrariness of $\mu_i(\mathbf{W})$, we can omit the integration over $\mathbf{W}$ in the above integrals and write for every $\mathbf{W}$:

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \big\{g_{[\mathbf{Q}]} \, , \, g^k(\boldsymbol{\theta}, \mathbf{W})\big\}\big|_{\mathcal{K}[0]} \; \frac{\delta\{J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[1]} \; \frac{d^m\theta}{(2\pi)^m} \; +$$

$$+ \int_0^{2\pi}\!\!\dots\int_0^{2\pi} \big\{J_{[\mathbf{p}]} \, , \, g^k(\boldsymbol{\theta}, \mathbf{W})\big\}\big|_{\mathcal{K}[1]} \; \frac{\delta\{g_{[\mathbf{Q}]} \, , \, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[0]} \; \frac{d^m\theta}{(2\pi)^m} \; -$$

$$- \int_0^{2\pi}\!\!\dots\int_0^{2\pi} \big\{J_{[\mathbf{q}]} \, , \, g^k(\boldsymbol{\theta}, \mathbf{W})\big\}\big|_{\mathcal{K}[1]} \; \frac{\delta\{g_{[\mathbf{Q}]} \, , \, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[0]} \; \frac{d^m\theta}{(2\pi)^m} \;\; \equiv \;\; 0$$

Finally, using relations (3.40) and (3.67), we can write the above identity as

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \big\{g_{[\mathbf{Q}]} \, , \, g^k(\boldsymbol{\theta}, \mathbf{W})\big\}\big|_{\mathcal{K}[0]} \; \frac{\delta\{J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[1]} \; \frac{d^m\theta}{(2\pi)^m} \; -$$

$$- \int_0^{2\pi}\!\!\dots\int_0^{2\pi} A^k_{[1][\mathbf{p}]}\left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W}\right) \; \frac{\delta\{g_{[\mathbf{Q}]} \, , \, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[0]} \; \frac{d^m\theta}{(2\pi)^m} \; +$$

$$+ \int_0^{2\pi}\!\!\dots\int_0^{2\pi} A^k_{[1][\mathbf{q}]}\left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W}\right) \; \frac{\delta\{g_{[\mathbf{Q}]} \, , \, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[0]} \; \frac{d^m\theta}{(2\pi)^m} \;\; \equiv \;\; 0$$

where the functions $A^k_{[1][\mathbf{q}]}(\boldsymbol{\theta}, \mathbf{W})$ are introduced by formula (3.43).

Now assume that the values $A^k_{[1][\mathbf{q}]}(\boldsymbol{\theta}, \mathbf{W})$ for $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$, according to (3.41), can be represented in the form

$$A^k_{[1][\mathbf{q}]}(\boldsymbol{\theta}, \mathbf{W}) \;\; = \;\; - \, \hat{B}^{kj}_{[0]}(\mathbf{W}) \; B_{j[\mathbf{q}](1)}(\boldsymbol{\theta}, \mathbf{W}) \tag{3.68}$$

with some smooth in $\boldsymbol{\theta}$, 2π-periodic in each θ^α functions $B_{j[\mathbf{q}](1)}(\boldsymbol{\theta}, \mathbf{W})$.

We can then write for $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$:

$$\begin{aligned} &\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \big\{g_{[\mathbf{Q}]} \, , \, g^k(\boldsymbol{\theta}, \mathbf{W})\big\}\big|_{\mathcal{K}[0]} \; \frac{\delta\{J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[1]} \; \frac{d^m\theta}{(2\pi)^m} \; + \\ &+ \int_0^{2\pi}\!\!\dots\int_0^{2\pi} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \; B_{j[\mathbf{p}](1)}\left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W}\right)\right] \; \frac{\delta\{g_{[\mathbf{Q}]} \, , \, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[0]} \; \frac{d^m\theta}{(2\pi)^m} \; - \\ &- \int_0^{2\pi}\!\!\dots\int_0^{2\pi} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \; B_{j[\mathbf{q}](1)}\left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W}\right)\right] \; \frac{\delta\{g_{[\mathbf{Q}]} \, , \, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[0]} \; \frac{d^m\theta}{(2\pi)^m} \;\; \equiv \;\; 0 \end{aligned} \tag{3.69}$$

where

$$\hat{B}^{ij}_{[0][\mathbf{S}]}(\mathbf{W}) \;\equiv\; \sum_{l_1,\dots,l_d} B^{ij}_{(l_1\dots l_d)} \left(\mathbf{\Phi}\left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta},\, \mathbf{S}_{\mathbf{W}}, U^1(\mathbf{W}), \dots, U^{m+s}(\mathbf{W}) \right), \,\dots \right) \;\times$$

$$\times\;\; k_1^{\alpha_1^1}(\mathbf{W}) \dots k_1^{\alpha_{l_1}^1}(\mathbf{W}) \;\dots\; k_d^{\alpha_1^d}(\mathbf{W}) \dots k_d^{\alpha_{l_d}^d}(\mathbf{W}) \;\; \frac{\partial}{\partial\theta^{\alpha_1^1}} \cdots \frac{\partial}{\partial\theta^{\alpha_{l_1}^1}} \;\cdots\; \frac{\partial}{\partial\theta^{\alpha_1^d}} \cdots \frac{\partial}{\partial\theta^{\alpha_{l_d}^d}}$$

Using expression (3.39) for the bracket of constraints on $\mathcal{K}$, as well as relations (3.21), (3.56), (3.62), and the skew-symmetry of the operator $\hat{B}^{ij}_{[0][\mathbf{S}]}(\mathbf{W})$, we obtain also the following relation:

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \left\{ g_{[\mathbf{Q}]} \,,\, g^k(\boldsymbol{\theta}, \mathbf{W}) \right\}\Big|_{\mathcal{K}[0]} \; \frac{\delta\{J_{[\mathbf{q}]} \,,\, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[1]} \; \frac{d^m\theta}{(2\pi)^m} \;=$$

$$= \;-\; \int_0^{2\pi}\!\!\dots\int_0^{2\pi} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \; Q_j \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \; \frac{\delta\{J_{[\mathbf{q}]} \,,\, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[1]} \; \frac{d^m\theta}{(2\pi)^m}$$

Expanding also the remaining terms of identity (3.69) according to (3.64), we get for $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$:

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \; Q_j \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \; \frac{\delta\{J_{[\mathbf{q}]} \,,\, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})}\bigg|_{\mathcal{K}[1]} \; \frac{d^m\theta}{(2\pi)^m} \;- \tag{3.70}$$

$$-\; \int_0^{2\pi}\!\!\dots\int_0^{2\pi} \frac{d^m\theta}{(2\pi)^m} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \; B_{j[\mathbf{p}](1)} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \;\times$$

$$\times \Bigg[\sum_{l_1,\dots,l_d} (-1)^{l_1+\dots+l_d} \; k_1^{\alpha_1^1}(\mathbf{W}) \dots k_1^{\alpha_{l_1}^1}(\mathbf{W}) \;\dots\; k_d^{\alpha_1^d}(\mathbf{W}) \dots k_d^{\alpha_{l_d}^d}(\mathbf{W}) \;\times$$

$$\times\; \frac{\partial}{\partial\theta^{\alpha_1^1}} \cdots \frac{\partial}{\partial\theta^{\alpha_{l_1}^1}} \;\cdots\; \frac{\partial}{\partial\theta^{\alpha_1^d}} \cdots \frac{\partial}{\partial\theta^{\alpha_{l_d}^d}} \; \left[Q_i \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) S^{i\mu}_{k(l_1\dots l_d)} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{W}), \dots \right)\Big|_{\mathcal{K}[0]} \right] q_\mu(\mathbf{W}) \;+$$

$$+\; \omega^{\alpha\mu}(\mathbf{W})\, q_\mu(\mathbf{W})\, Q_{k,\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \Bigg] \;+$$

$$+\; \int_0^{2\pi}\!\!\dots\int_0^{2\pi} \frac{d^m\theta}{(2\pi)^m} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \; B_{j[\mathbf{q}](1)} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \;\times$$

$$\times \Bigg[\sum_{l_1,\dots,l_d} (-1)^{l_1+\dots+l_d} \; k_1^{\alpha_1^1}(\mathbf{W}) \dots k_1^{\alpha_{l_1}^1}(\mathbf{W}) \;\dots\; k_d^{\alpha_1^d}(\mathbf{W}) \dots k_d^{\alpha_{l_d}^d}(\mathbf{W}) \;\times$$

$$\times\; \frac{\partial}{\partial\theta^{\alpha_1^1}} \cdots \frac{\partial}{\partial\theta^{\alpha_{l_1}^1}} \;\cdots\; \frac{\partial}{\partial\theta^{\alpha_1^d}} \cdots \frac{\partial}{\partial\theta^{\alpha_{l_d}^d}} \; \left[Q_i \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) S^{i\mu}_{k(l_1\dots l_d)} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{W}), \dots \right)\Big|_{\mathcal{K}[0]} \right] p_\mu(\mathbf{W}) \;+$$

$$+\; \omega^{\alpha\mu}(\mathbf{W})\, p_\mu(\mathbf{W})\, Q_{k,\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \Bigg] \;\equiv\; 0$$

Consider now the Jacobi identity of the form

$$\{g_{[\mathbf{P}]}\,,\,\{g_{[\mathbf{Q}]}\,,\,J_{[\mathbf{q}]}\}\} \,+\, \{g_{[\mathbf{Q}]}\,,\,\{J_{[\mathbf{q}]}\,,\,g_{[\mathbf{P}]}\}\} \,+\, \{J_{[\mathbf{q}]}\,,\,\{g_{[\mathbf{P}]}\,,\,g_{[\mathbf{Q}]}\}\} \;\equiv\; 0 \tag{3.71}$$

for arbitrary fixed functions $\mathbf{q}(\mathbf{X})$, and the functionals $\mathbf{P}(\boldsymbol{\theta},\mathbf{X})$ and $\mathbf{Q}(\boldsymbol{\theta},\mathbf{X})$ defined as before with the aid of arbitrary functions $\tilde{\mathbf{P}}(\boldsymbol{\theta},\mathbf{X})$, $\tilde{\mathbf{Q}}(\boldsymbol{\theta},\mathbf{X})$.

According to the relations

$$\{g_{[\mathbf{Q},\mathbf{P}]}\,,\,U^{\mu}(\mathbf{W})\}\big|_{\mathcal{K}} \;=\; O(\epsilon) \quad , \quad \{J_{[\mathbf{q}]}\,,\,g^{k}(\boldsymbol{\theta},\mathbf{W})\}\big|_{\mathcal{K}} \;=\; O(\epsilon) \quad , \quad \{J_{[\mathbf{q}]}\,,\,U^{\mu}(\mathbf{W})\}\big|_{\mathcal{K}} \;=\; O(\epsilon)$$

it's not difficult to see that after the restriction on $\mathcal{K}$ the major term (in ϵ) of (3.71) will be written as

$$\int \{g_{[\mathbf{P}]}\,,\,g^{k}(\boldsymbol{\theta},\mathbf{W})\}\big|_{\mathcal{K}[0]}\;\frac{\delta\{g_{[\mathbf{Q}]}\,,\,J_{[\mathbf{q}]}\}}{\delta g^{k}(\boldsymbol{\theta},\mathbf{W})}\bigg|_{\mathcal{K}[0]}\;\frac{d^{m}\theta}{(2\pi)^{m}}\,d^{d}W \;-$$

$$-\;\int \{g_{[\mathbf{Q}]}\,,\,g^{k}(\boldsymbol{\theta},\mathbf{W})\}\big|_{\mathcal{K}[0]}\;\frac{\delta\{g_{[\mathbf{P}]}\,,\,J_{[\mathbf{q}]}\}}{\delta g^{k}(\boldsymbol{\theta},\mathbf{W})}\bigg|_{\mathcal{K}[0]}\;\frac{d^{m}\theta}{(2\pi)^{m}}\,d^{d}W \;\;\equiv\;\; 0$$

Again recalling that $q_{\nu}(\mathbf{W})$ are arbitrary functions of $\mathbf{W}$ appearing in the integrand in the form of local factors, we can write the above relation in a stronger form. That is, for every $\mathbf{W}$

$$\begin{aligned}
&\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi}\;\{g_{[\mathbf{P}]}\,,\,g^{k}(\boldsymbol{\theta},\mathbf{W})\}\big|_{\mathcal{K}[0]}\;\frac{\delta\{g_{[\mathbf{Q}]}\,,\,J_{[\mathbf{q}]}\}}{\delta g^{k}(\boldsymbol{\theta},\mathbf{W})}\bigg|_{\mathcal{K}[0]}\;\frac{d^{m}\theta}{(2\pi)^{m}} \;\;- \\
-\;&\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi}\;\{g_{[\mathbf{Q}]}\,,\,g^{k}(\boldsymbol{\theta},\mathbf{W})\}\big|_{\mathcal{K}[0]}\;\frac{\delta\{g_{[\mathbf{P}]}\,,\,J_{[\mathbf{q}]}\}}{\delta g^{k}(\boldsymbol{\theta},\mathbf{W})}\bigg|_{\mathcal{K}[0]}\;\frac{d^{m}\theta}{(2\pi)^{m}} \;\;\equiv\;\; 0
\end{aligned} \tag{3.72}$$

As well as in the case of identity (3.70), using relations (3.39), (3.21), (3.56), (3.67), we can write identity (3.72) in the form:

$$\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi}\frac{d^{m}\theta}{(2\pi)^{m}}\left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W})\;P_{j}\left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon}+\boldsymbol{\theta},\mathbf{W}\right)\right]\;\times \tag{3.73}$$

$$\begin{aligned}
\times&\left[\sum_{l_1,\dots,l_d}(-1)^{l_1+\dots+l_d}\;k_{1}^{\alpha_1^1}(\mathbf{W})\dots k_{1}^{\alpha_{l_1}^1}(\mathbf{W})\;\dots\;k_{d}^{\alpha_1^d}(\mathbf{W})\dots k_{d}^{\alpha_{l_d}^d}(\mathbf{W})\;\times\right.\\
\times&\;\frac{\partial}{\partial\theta^{\alpha_1^1}}\dots\frac{\partial}{\partial\theta^{\alpha_{l_1}^1}}\;\dots\;\frac{\partial}{\partial\theta^{\alpha_1^d}}\dots\frac{\partial}{\partial\theta^{\alpha_{l_d}^d}}\;\left[Q_{i}\left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon}+\boldsymbol{\theta},\mathbf{W}\right)S^{i\mu}_{k(l_1\dots l_d)}\left(\boldsymbol{\varphi}(\boldsymbol{\theta},\mathbf{W}),\dots\right)\Big|_{\mathcal{K}[0]}\right]q_{\mu}(\mathbf{W})\;+\\
&\left.+\;\omega^{\alpha\mu}(\mathbf{W})\,q_{\mu}(\mathbf{W})\,Q_{k,\theta^{\alpha}}\left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon}+\boldsymbol{\theta},\mathbf{W}\right)\right]-
\end{aligned}$$

$$-\;\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi}\frac{d^{m}\theta}{(2\pi)^{m}}\left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W})\;Q_{j}\left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon}+\boldsymbol{\theta},\mathbf{W}\right)\right]\;\times$$

$$\times \left[\sum_{l_1,\dots,l_d} (-1)^{l_1+\dots+l_d} \; k_1^{\alpha_1^1}(\mathbf{W}) \dots k_1^{\alpha_{l_1}^1}(\mathbf{W}) \; \dots \; k_d^{\alpha_1^d}(\mathbf{W}) \dots k_d^{\alpha_{l_d}^d}(\mathbf{W}) \; \times \right.$$

$$\times \; \frac{\partial}{\partial\theta^{\alpha_1^1}} \dots \frac{\partial}{\partial\theta^{\alpha_{l_1}^1}} \; \dots \; \frac{\partial}{\partial\theta^{\alpha_1^d}} \dots \frac{\partial}{\partial\theta^{\alpha_{l_d}^d}} \; \left[P_i \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) S^{i\mu}_{k(l_1\dots l_d)} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{W}), \dots \right) \Big|_{\mathcal{K}[0]} \right] q_\mu(\mathbf{W}) \; +$$

$$\left. + \; \omega^{\alpha\mu}(\mathbf{W}) \, q_\mu(\mathbf{W}) \, P_{k,\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \;\equiv\; 0$$

Note now that the values of $\mathbf{Q}(\boldsymbol{\theta}, \mathbf{X})$ and $\mathbf{P}(\boldsymbol{\theta}, \mathbf{X})$ are arbitrary 2π-periodic functions of $\boldsymbol{\theta}$, satisfying conditions (3.56). In particular, we can put in (3.73) at $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$

$$\mathbf{P}(\boldsymbol{\theta}, \mathbf{W}) \;=\; \mathbf{B}_{[\mathbf{p}](1)}(\boldsymbol{\theta}, \mathbf{W}) \quad \text{ or } \quad \mathbf{P}(\boldsymbol{\theta}, \mathbf{W}) \;=\; \mathbf{B}_{[\mathbf{q}](1)}(\boldsymbol{\theta}, \mathbf{W}) \tag{3.74}$$

By analogy with (3.64) we introduce for convenience the notation for $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$:

$$\frac{\delta \{ g_{[\epsilon \mathbf{B}_{[\mathbf{p}](1)}]} \, , \, J_{[\mathbf{q}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}[1]} \;\equiv \tag{3.75}$$

$$\equiv \left[\sum_{l_1,\dots,l_d} (-1)^{l_1+\dots+l_d} \; k_1^{\alpha_1^1}(\mathbf{W}) \dots k_1^{\alpha_{l_1}^1}(\mathbf{W}) \; \dots \; k_d^{\alpha_1^d}(\mathbf{W}) \dots k_d^{\alpha_{l_d}^d}(\mathbf{W}) \; \times \right.$$

$$\times \; \frac{\partial}{\partial\theta^{\alpha_1^1}} \dots \frac{\partial}{\partial\theta^{\alpha_{l_1}^1}} \; \dots \; \frac{\partial}{\partial\theta^{\alpha_1^d}} \dots \frac{\partial}{\partial\theta^{\alpha_{l_d}^d}} \; \left[B_{i[\mathbf{p}](1)} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) S^{i\mu}_{k(l_1\dots l_d)} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{W}), \dots \right) \Big|_{\mathcal{K}[0]} \right] q_\mu(\mathbf{W}) \; +$$

$$\left. + \; \omega^{\alpha\mu}(\mathbf{W}) \, q_\mu(\mathbf{W}) \, B_{k[\mathbf{p}](1),\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right]$$

for arbitrary smooth functions $\mathbf{q}(\mathbf{X})$, $\mathbf{p}(\mathbf{X})$.

Note that the functional $g_{[\epsilon \mathbf{B}_{[\mathbf{p}](1)}]}$ is not defined on the whole functional space, so relation (3.75) plays just a role of a formal notation for $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$.

Using now (3.73) for the functions (3.74), we can rewrite (3.70) in the form

$$\int_0^{2\pi} \!\!\! \dots \int_0^{2\pi} \frac{d^m\theta}{(2\pi)^m} \; \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \; Q_j \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \; \times$$

$$\times \; \left(\frac{\delta \{ J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}[1]} - \; \frac{\delta \{ g_{[\epsilon \mathbf{B}_{[\mathbf{p}](1)}]} \, , \, J_{[\mathbf{q}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}[1]} + \; \frac{\delta \{ g_{[\epsilon \mathbf{B}_{[\mathbf{q}](1)}]} \, , \, J_{[\mathbf{p}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}[1]} \right) \;\equiv\; 0$$

provided that $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$.

Using the skew-symmetry of the Hamiltonian operator $\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W})$ on Λ we can then write

$$\int_0^{2\pi} \!\!\! \dots \int_0^{2\pi} \frac{d^m\theta}{(2\pi)^m} \, Q_j \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \; \times$$

$$\times \; \left[\hat{B}^{jk}_{[0][\mathbf{S}]}(\mathbf{W}) \; \left(\frac{\delta \{ J_{[\mathbf{q}]} \, , \, J_{[\mathbf{p}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}[1]} - \; \frac{\delta \{ g_{[\epsilon \mathbf{B}_{[\mathbf{p}](1)}]} \, , \, J_{[\mathbf{q}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}[1]} + \; \frac{\delta \{ g_{[\epsilon \mathbf{B}_{[\mathbf{q}](1)}]} \, , \, J_{[\mathbf{p}]} \}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \Bigg|_{\mathcal{K}[1]} \right) \right] \;\equiv\; 0$$

The values $Q_j(\boldsymbol{\theta}, \mathbf{W})$ are arbitrary smooth 2π-periodic functions of $\boldsymbol{\theta}$ with the only restriction (3.56). We know also that the value in the brackets is a smooth 2π-periodic in each θ^α function of $\boldsymbol{\theta}$ for $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$. As a consequence, we can write for $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$

$$\hat{B}^{jk}_{[0][\mathbf{S}]}(\mathbf{W}) \left(\left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} - \left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{p}](1)}]}, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} + \left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{q}](1)}]}, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \right) \equiv$$

$$\equiv \sum_{\alpha=1}^{m} a^\alpha_{[\mathbf{q},\mathbf{p}]}(\mathbf{U}(\mathbf{W}), \mathbf{U}_{\mathbf{W}}(\mathbf{W})) \; \Phi^j_{\theta^\alpha} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{W}) \right)$$

with some coefficients $a^\alpha_{[\mathbf{q},\mathbf{p}]}(\mathbf{U}(\mathbf{W}), \mathbf{U}_{\mathbf{W}}(\mathbf{W}))$.

The values in parentheses are smooth 2π-periodic functions of $\boldsymbol{\theta}$ at $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$. According to Lemma 3.6 we can therefore say that up to a linear combination of the regular eigenvectors $\mathbf{v}^{(l)}_{[\mathbf{U}(\mathbf{W})]}(\mathbf{S}(\mathbf{W})/\epsilon + \boldsymbol{\theta})$ of $\hat{B}^{jk}_{[0][\mathbf{S}]}(\mathbf{W}) = \hat{B}^{jk}_{[0][\mathbf{S}]}(\mathbf{U}(\mathbf{W}))$, corresponding to zero eigenvalues, the value in parentheses is a linear combination of the variation derivatives (2.10), generating linear shifts of the phases on Λ. For a complete Hamiltonian set of the integrals $(I^1, \dots, I^N)$ we can then write according to (2.11) and (3.54)

$$\left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} - \left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{p}](1)}]}, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} + \left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{q}](1)}]}, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \equiv$$

$$\equiv \sum_{q=1}^{m+s} b_{q[\mathbf{q},\mathbf{p}]}(\mathbf{U}(\mathbf{W}), \mathbf{U}_{\mathbf{W}}(\mathbf{W})) \; \kappa^{(q)}_{k[\mathbf{U}(\mathbf{W})]} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta} \right) \tag{3.76}$$

with some coefficients $b_{q[\mathbf{q},\mathbf{p}]}(\mathbf{U}(\mathbf{W}), \mathbf{U}_{\mathbf{W}}(\mathbf{W}))$ at $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$.

Consider now the Jacobi identity of the form

$$\left\{ \left\{ J_{[\mathbf{q}]}, J_{[\mathbf{p}]} \right\}, J_{[\mathbf{r}]} \right\} \; + \; \left\{ \left\{ J_{[\mathbf{p}]}, J_{[\mathbf{r}]} \right\}, J_{[\mathbf{q}]} \right\} \; + \; \left\{ \left\{ J_{[\mathbf{r}]}, J_{[\mathbf{q}]} \right\}, J_{[\mathbf{p}]} \right\} \; \equiv \; 0$$

with arbitrary smooth functions $\mathbf{q}(\mathbf{X})$, $\mathbf{p}(\mathbf{X})$, $\mathbf{r}(\mathbf{X})$.

In the main ($\sim \epsilon^2$) order on $\mathcal{K}$ the given identity leads to the relations

$$\int \left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta S^\alpha(\mathbf{W})} \right|_{\mathcal{K}[1]} \left. \left\{ S^\alpha(\mathbf{W}) \, , \, J_{[\mathbf{r}]} \right\} \right|_{\mathcal{K}[1]} d^d W \quad + \quad c.p. \quad +$$

$$+ \quad \int \left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta U^\gamma(\mathbf{W})} \right|_{\mathcal{K}[1]} \left. \left\{ U^\gamma(\mathbf{W}) \, , \, J_{[\mathbf{r}]} \right\} \right|_{\mathcal{K}[1]} d^d W \quad + \quad c.p. \quad + \tag{3.77}$$

$$+ \int\!\!\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \left. \left\{ g^k(\boldsymbol{\theta}, \mathbf{W}) \, , \, J_{[\mathbf{r}]} \right\} \right|_{\mathcal{K}[1]} \frac{d^m \theta}{(2\pi)^m} \, d^d W \; + \; c.p. \; \equiv \; 0$$

$(\alpha = 1, \dots, m, \;\; \gamma = 1, \dots, m+s)$.

Again, using relations (3.40), (3.43) and (3.62), we can replace identity (3.77) by the following relation

$$\int \left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta S^\alpha(\mathbf{W})} \right|_{\mathcal{K}[1]} \left. \left\{ S^\alpha(\mathbf{W}) \, , \, J_{[\mathbf{r}]} \right\} \right|_{\mathcal{K}[1]} d^d W \quad + \quad c.p. \quad +$$

$$+ \int \left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta U^{\gamma}(\mathbf{W})} \right|_{\mathcal{K}[1]} \left\{ U^{\gamma}(\mathbf{W}) \, , \, J_{[\mathbf{r}]} \right\} \Big|_{\mathcal{K}[1]} \, d^d W \quad + \quad c.p. \quad + \tag{3.78}$$

$$+ \iint_0^{2\pi} \!\! \dots \int_0^{2\pi} \left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} A^k_{[1][\mathbf{r}]} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \frac{d^m \theta}{(2\pi)^m} \, d^d W \quad + \quad c.p. \quad \equiv \quad 0$$

Using relations (3.44) and the representations (3.68) and (3.76), we can write for $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$:

$$\int_0^{2\pi} \!\! \dots \int_0^{2\pi} \left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} A^k_{[1][\mathbf{r}]} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \frac{d^m \theta}{(2\pi)^m} \quad + \quad c.p. \quad = \tag{3.79}$$

$$= \int_0^{2\pi} \!\! \dots \int_0^{2\pi} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \, B_{j[\mathbf{r}](1)} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \left[\left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{q}](1)}]}, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} - \left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{p}](1)}]}, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \right] \frac{d^m \theta}{(2\pi)^m} +$$

$$+ \int_0^{2\pi} \!\! \dots \int_0^{2\pi} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \, B_{j[\mathbf{q}](1)} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \left[\left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{p}](1)}]}, J_{[\mathbf{r}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} - \left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{r}](1)}]}, J_{[\mathbf{p}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \right] \frac{d^m \theta}{(2\pi)^m} +$$

$$+ \int_0^{2\pi} \!\! \dots \int_0^{2\pi} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \, B_{j[\mathbf{p}](1)} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \left[\left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{r}](1)}]}, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} - \left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{q}](1)}]}, J_{[\mathbf{r}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \right] \frac{d^m \theta}{(2\pi)^m}$$

Similarly to earlier arguments, substituting now in identity (3.73):

$$\mathbf{Q}(\boldsymbol{\theta}, \mathbf{W}) \; = \; \mathbf{B}_{[\mathbf{r}](1)}(\boldsymbol{\theta}, \mathbf{W}) \quad , \quad \mathbf{P}(\boldsymbol{\theta}, \mathbf{W}) \; = \; \mathbf{B}_{[\mathbf{p}](1)}(\boldsymbol{\theta}, \mathbf{W})$$

for $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$ we obtain the identities

$$\int_0^{2\pi} \!\! \dots \int_0^{2\pi} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \; B_{j[\mathbf{p}](1)} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{r}](1)}]}, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \frac{d^m \theta}{(2\pi)^m} \; - \tag{3.80}$$

$$- \int_0^{2\pi} \!\! \dots \int_0^{2\pi} \left[\hat{B}^{kj}_{[0][\mathbf{S}]}(\mathbf{W}) \; B_{j[\mathbf{r}](1)} \left(\frac{\mathbf{S}(\mathbf{W})}{\epsilon} + \boldsymbol{\theta}, \mathbf{W} \right) \right] \left. \frac{\delta\{g_{[\epsilon \mathbf{B}_{[\mathbf{p}](1)}]}, J_{[\mathbf{q}]}\}}{\delta g^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} \frac{d^m \theta}{(2\pi)^m} \quad = \quad 0$$

Using the cyclic permutations of the functions $\mathbf{q}(\mathbf{X})$, $\mathbf{p}(\mathbf{X})$, and $\mathbf{r}(\mathbf{X})$ in identity (3.80), it's not difficult to see that the right-hand part of relation (3.79) is identically equal to zero at $\mathbf{U}(\mathbf{W}) \in \mathcal{S}$. It's not difficult to see also that the left-hand side of the relation (3.79) is a smooth regular function of the parameters $\mathbf{U}(\mathbf{W}) = (\mathbf{S}_{\mathbf{W}}, U^1(\mathbf{W}), \dots, U^{m+s}(\mathbf{W}))$ and their derivatives. Using the fact that the set $\mathcal{S}$ is everywhere dense in the parameter space $\mathbf{U}$, we can conclude that the left-hand side of equation (3.79) is identically equal to zero under the conditions of the theorem.

We have, therefore, that under the conditions of the theorem, the identity (3.77) implies the relations

$$\int \left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta S^{\alpha}(\mathbf{W})} \right|_{\mathcal{K}[1]} \left\{ S^{\alpha}(\mathbf{W}) \, , \, J_{[\mathbf{r}]} \right\} \Big|_{\mathcal{K}[1]} \, d^d W \quad + \quad c.p. \quad +$$

$$+ \int \left. \frac{\delta\{J_{[\mathbf{q}]}, J_{[\mathbf{p}]}\}}{\delta U^{\gamma}(\mathbf{W})} \right|_{\mathcal{K}[1]} \left\{ U^{\gamma}(\mathbf{W}) \, , \, J_{[\mathbf{r}]} \right\} \Big|_{\mathcal{K}[1]} \, d^d W \quad + \quad c.p. \quad \equiv \quad 0$$

Using the relations

$$\left.\frac{\delta\{J_{[\mathbf{q}]},J_{[\mathbf{p}]}\}}{\delta S^{\alpha}(\mathbf{W})}\right|_{\mathcal{K}[1]} = \frac{\delta\{U_{[\mathbf{q}]},U_{[\mathbf{p}]}\}_{DN}}{\delta S^{\alpha}(\mathbf{W})} \quad , \quad \left.\frac{\delta\{J_{[\mathbf{q}]},J_{[\mathbf{p}]}\}}{\delta U^{\gamma}(\mathbf{W})}\right|_{\mathcal{K}[1]} = \frac{\delta\{U_{[\mathbf{q}]},U_{[\mathbf{p}]}\}_{DN}}{\delta U^{\gamma}(\mathbf{W})}$$

$$\left\{U^{\gamma}(\mathbf{W}),J_{[\mathbf{r}]}\right\}\big|_{\mathcal{K}[1]} = \left\{U^{\gamma}(\mathbf{W}),U_{[\mathbf{r}]}\right\}_{DN}$$

we get then the identities

$$\int \frac{\delta\{U_{[\mathbf{q}]},U_{[\mathbf{p}]}\}_{DN}}{\delta S^{\alpha}(\mathbf{W})} \left\{S^{\alpha}(\mathbf{W}),U_{[\mathbf{r}]}\right\}_{DN} d^{d}W \quad + \quad c.p. \ + $$
$$+ \int \frac{\delta\{U_{[\mathbf{q}]},U_{[\mathbf{p}]}\}_{DN}}{\delta U^{\gamma}(\mathbf{W})} \left\{U^{\gamma}(\mathbf{W}),U_{[\mathbf{r}]}\right\}_{DN} d^{d}W \quad + \quad c.p. \ \equiv \ 0 \quad (3.81)$$

(summation in $\alpha = 1,\dots,m$, $\gamma = 1,\dots,m+s$).

Here the notation $\{S^{\alpha}(\mathbf{W}),U_{[\mathbf{r}]}\}_{DN}$ means the pairing of $S^{\alpha}(\mathbf{W})$ as the functional of $\mathbf{U}(\mathbf{Z})$ with $U_{[\mathbf{r}]}$ given by the Dubrovin - Novikov skew-symmetric form and coinciding with the corresponding value for the bracket (3.55). Easy to see then that relations (3.81) give in particular the Jacobi identities for the bracket (3.55) on the space of fields $(S^{\alpha}(\mathbf{X}),U^{1}(\mathbf{X}),\dots,U^{m+s}(\mathbf{X}))$.

Theorem 3.1 is proved.

We will not consider here applications of Theorem 3.1 in details. The example of the application of Theorem 3.1 in one-dimensional case can be found in [25]. In general, we can say that Theorem 3.1 gives justification of the procedure of the bracket averaging for a wide class of systems having multi-phase solutions.

Using Lemma 3.5 we can formulate now the theorem that gives the justification of the procedure of the bracket averaging for a regular Hamiltonian family of single-phase solutions of system (2.5).

Theorem 3.1′.

Let Λ be a regular Hamiltonian family of single-phase solutions of system (2.5). Let $(I^{1},\dots,I^{N})$ be a complete Hamiltonian set of commuting first integrals of system (2.5) on Λ having the form (2.8). Then the bracket

$$\{S(\mathbf{X})\,,\,S(\mathbf{Y})\} \ = \ 0 \quad , \qquad \{S(\mathbf{X})\,,\,U^{\gamma}(\mathbf{Y})\} \ = \ \omega^{\gamma}\left(S_{\mathbf{X}},\,U^{1}(\mathbf{X}),\dots,U^{s+1}(\mathbf{X})\right)\ \delta(\mathbf{X}-\mathbf{Y})\ ,$$

$$\{U^{\gamma}(\mathbf{X})\,,\,U^{\rho}(\mathbf{Y})\} \ = \ \langle A^{\gamma\rho}_{10\dots0}\rangle\left(S_{\mathbf{X}},\,U^{1}(\mathbf{X}),\dots,U^{s+1}(\mathbf{X})\right)\ \delta_{X^{1}}(\mathbf{X}-\mathbf{Y}) \ + \ \dots \ +$$

$$+ \ \langle A^{\gamma\rho}_{0\dots01}\rangle\left(S_{\mathbf{X}},\,U^{1}(\mathbf{X}),\dots,U^{s+1}(\mathbf{X})\right)\ \delta_{X^{d}}(\mathbf{X}-\mathbf{Y}) \ +$$

$$+ \ \left[\langle Q^{\gamma\rho p}\rangle\left(S_{\mathbf{X}},\,U^{1}(\mathbf{X}),\dots,U^{s+1}(\mathbf{X})\right)\right]_{X^{p}}\ \delta(\mathbf{X}-\mathbf{Y}) \quad , \qquad \gamma,\rho \ = \ 1,\dots,s+1 \quad (3.82)$$

on the space of fields $(S(\mathbf{X})\,,\,U^{\gamma}(\mathbf{X}))$, $\gamma \ = \ 1,\dots,s+1$, satisfies the Jacobi identity.

Let us prove now the second theorem justifying the invariance of the bracket averaging procedure.

Theorem 3.2.

Let Λ be a regular Hamiltonian family of m-phase solutions of (2.5). Let $(I^1, \dots, I^N)$ and $(I'^1, \dots, I'^N)$ be two different complete Hamiltonian sets of commuting first integrals of (2.5) having the form (2.8). Then the brackets (3.55) obtained using the sets $(I^1, \dots, I^N)$ and $(I'^1, \dots, I'^N)$ coincide with each other.

Proof.

Note that the sets $(I^1, \dots, I^N)$, $(I'^1, \dots, I'^N)$ correspond to two different systems of coordinates $(U^1, \dots, U^N)$, $(U'^1, \dots, U'^N)$ on the family Λ, given by the averages of the functionals $\mathbf{I}$ and $\mathbf{I}'$. Let us first prove that the Dubrovin - Novikov form obtained using the set $(I'^1, \dots, I'^N)$ coincides with the form, obtained using the set $(I^1, \dots, I^N)$, after the corresponding change of coordinates.

$$U'^{\nu} \; = \; U'^{\nu}(\mathbf{U})$$

Consider the functionals

$$J^{\nu}(\mathbf{X}) \; = \; \int_0^{2\pi} \!\! \dots \int_0^{2\pi} P^{\nu}(\boldsymbol{\varphi}, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \dots) \, \frac{d^m \theta}{(2\pi)^m} \quad , \quad \nu = 1, \dots, N$$

Consider now the values of the functionals $S^{\alpha}(\mathbf{X})$, $U^{\gamma}(\mathbf{X})$, and the constraints $g^i(\boldsymbol{\theta}, \mathbf{X})$, introduced with the aid of the functionals $\mathbf{J}(\mathbf{X})$, as a coordinate system in the neighborhood of the submanifold $\mathcal{K}$.

We obviously have the relations

$$J^{\nu}(\mathbf{X})|_{\mathcal{K}} \; = \; U^{\nu}(\mathbf{X}) \, + \, O(\epsilon) \quad , \quad J'^{\nu}(\mathbf{X})|_{\mathcal{K}} \; = \; U'^{\nu}(\mathbf{X}) \, + \, O(\epsilon)$$

on the submanifold $\mathcal{K}$.

For values of the functionals $J'^{\nu}(\mathbf{X})$ on the submanifold $\mathcal{K}$ we can then write the relations

$$J'^{\nu}(\mathbf{X})|_{\mathcal{K}} \; = \; U'^{\nu} \left(\mathbf{J}(\mathbf{X}) \right) \; + \; \sum_{l \geq 1} \epsilon^l \, j'^{\nu}_{(l)} \left(\mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}), \dots \right)$$

where $j'^{\nu}_{(l)}$ are smooth functions of $(\mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}), \dots)$ and their derivatives, polynomial in the derivatives $(\mathbf{S}_{\mathbf{XX}}, \mathbf{U}_{\mathbf{X}}, \dots)$ and having degree l according to our previous definition.

Expanding the values of $J'^{\nu}(\mathbf{X})$ in the neighborhood of the submanifold $\mathcal{K}$, we can write

$$J'^{\nu}(\mathbf{X}) \; = \; U'^{\nu} \left(\mathbf{J}(\mathbf{X}) \right) \; + \; \sum_{l \geq 1} \epsilon^l \, j'^{\nu}_{(l)} \left(\mathbf{S}_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}), \dots \right) \; +$$
$$+ \; \int \int_0^{2\pi} \!\! \dots \int_0^{2\pi} T'^{\nu}_i(\mathbf{X}, \boldsymbol{\theta}, \mathbf{Y}, \epsilon) \; g^i(\boldsymbol{\theta}, \mathbf{Y}) \; \frac{d^m \theta}{(2\pi)^m} \, d^d Y \; + \; O(\mathbf{g}^2)$$

where, according to the form of constraints (3.11), we can put

$$T'^{\nu}_i(\mathbf{X}, \boldsymbol{\theta}, \mathbf{Y}, \epsilon) \; = \; \sum_{l_1, \dots, l_d} \Pi'^{\nu (l_1 \dots l_d)}_i \left(\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}), \epsilon \boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta}, \mathbf{X}), \dots \right) \Big|_{\mathcal{K}} \, \epsilon^{l_1 + \dots + l_d} \;\; \delta_{l_1 X^1 \dots l_d X^d}(\mathbf{X} - \mathbf{Y}) \; \equiv$$
$$\equiv \; \sum_{l_1, \dots, l_d} \frac{\partial P'^{\nu}}{\partial \varphi^i_{l_1 x^1 \dots l_d x^d}} \left(\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}), \epsilon \boldsymbol{\varphi}_{\mathbf{X}}(\boldsymbol{\theta}, \mathbf{X}), \dots \right) \Bigg|_{\mathcal{K}} \; \epsilon^{l_1 + \dots + l_d} \;\; \delta_{l_1 X^1 \dots l_d X^d}(\mathbf{X} - \mathbf{Y})$$

Considering the functionals $J'_{[\mathbf{q}]} = \int q_\nu(\mathbf{X})\, J'^{\nu}(\mathbf{X})\, d^d X$ with arbitrary smooth (compactly supported) functions $q_\nu(\mathbf{X})$, we can write in the vicinity of $\mathcal{K}$:

$$J'_{[\mathbf{q}]} \;=\; \int q_\nu(\mathbf{X}) \left[U'^{\nu}\left(\mathbf{J}(\mathbf{X})\right) \,+\, \sum_{l\geq 1} \epsilon^l\, j'^{\nu}_{(l)}\left(\mathbf{S_X}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}), \dots\right)\right] d^d X \,+ \tag{3.83}$$

$$+ \int \sum_{l_1,\dots,l_d} (-1)^{l_1+\dots+l_d}\; \epsilon^{l_1+\dots+l_d} \left[\frac{d^{l_1}}{dY_1^{l_1}} \dots \frac{d^{l_d}}{dY_d^{l_d}}\, q_\nu(\mathbf{Y})\; \Pi_i'^{\nu (l_1\dots l_d)}\left(\boldsymbol{\varphi}(\boldsymbol{\theta},\mathbf{Y}), \epsilon \boldsymbol{\varphi}_{\mathbf{Y}}(\boldsymbol{\theta},\mathbf{Y}), \dots\right)\Big|_{\mathcal{K}}\right] \times$$

$$\times\; g^i(\boldsymbol{\theta},\mathbf{Y})\; \frac{d^m\theta}{(2\pi)^m}\, d^d Y \quad + \quad O(\mathbf{g}^2)$$

The leading term (in ϵ) in the second part of expression (3.83) is given by the expression

$$\int q_\nu(\mathbf{Y}) \sum_{l_1,\dots,l_d} (-1)^{l_1+\dots+l_d}\; k_1^{\alpha^1_1}(\mathbf{Y}) \dots k_1^{\alpha^1_{l_1}}(\mathbf{Y}) \;\dots\; k_d^{\alpha^d_1}(\mathbf{Y}) \dots k_d^{\alpha^d_{l_d}}(\mathbf{Y}) \;\times$$

$$\times\; \Pi'^{\nu (l_1\dots l_d)}_{i,\, \theta^{\alpha^1_1}\dots \theta^{\alpha^1_{l_1}} \dots \theta^{\alpha^d_1}\dots \theta^{\alpha^d_{l_d}}} \left(\boldsymbol{\Phi}\left(\frac{\mathbf{S}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}, \mathbf{Y}\right), \dots \right)\; g^i(\boldsymbol{\theta},\mathbf{Y})\; \frac{d^m\theta}{(2\pi)^m}\; d^d Y$$

and coincides with the value

$$\int\!\! \int_0^{2\pi}\!\! \dots \int_0^{2\pi} q_\nu(\mathbf{Y})\, \zeta'^{(\nu)}_{i[\mathbf{U}(\mathbf{Y})]}\left(\frac{\mathbf{S}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}\right)\; g^i(\boldsymbol{\theta},\mathbf{Y})\; \frac{d^m\theta}{(2\pi)^m}\, d^d Y$$

where

$$\zeta'^{(\nu)}_{i[\mathbf{U}]}(\boldsymbol{\theta}) \;=\; \left[\frac{\delta}{\delta \varphi^i(\boldsymbol{\theta})} \int_0^{2\pi}\!\! \dots \int_0^{2\pi} P'^{\nu}(\boldsymbol{\varphi},\, k_1^{\beta_1} \boldsymbol{\varphi}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d} \boldsymbol{\varphi}_{\theta^{\beta_d}}, \dots)\; \frac{d^m\theta}{(2\pi)^m}\right]\Bigg|_{\boldsymbol{\varphi}(\boldsymbol{\theta}) = \boldsymbol{\Phi}(\boldsymbol{\theta},\mathbf{U})}$$

As the values $\zeta^{(\nu)}_{i[\mathbf{U}]}(\boldsymbol{\theta})$, the values $\zeta'^{(\nu)}_{i[\mathbf{U}]}(\boldsymbol{\theta})$ represent regular left eigenvectors of the operator $\hat{L}^i_{j[\mathbf{U}]}$ corresponding to the zero eigenvalue. In the case of a complete regular family of m-phase solutions, we have therefore

$$\zeta'^{(\nu)}_{i[\mathbf{U}]}(\boldsymbol{\theta}) \;=\; \sum_q \Gamma^\nu_q(\mathbf{U})\; \kappa^{(q)}_{i[\mathbf{U}]}(\boldsymbol{\theta})$$

for some functions $\Gamma^\nu_q(\mathbf{U})$.

We can write, therefore, up to quadratic terms in $\mathbf{g}(\boldsymbol{\theta},\mathbf{X})$:

$$\int q_\nu(\mathbf{X})\, J'^{\nu}(\mathbf{X})\, d^d X \;=\; \int q_\nu(\mathbf{X}) \left[U'^{\nu}\left(\mathbf{J}(\mathbf{X})\right) \,+\, \sum_{l\geq 1} \epsilon^l\, j'^{\nu}_{(l)}\left(\mathbf{S_X}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}), \dots\right)\right] d^d X \,+$$

$$+ \int q_\nu(\mathbf{X}) \left[\sum_q \Gamma^\nu_q(\mathbf{U})\; \kappa^{(q)}_{i[\mathbf{U}(\mathbf{X})]}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}\right) \,+\, O(\epsilon)\right] g^i(\boldsymbol{\theta},\mathbf{X})\; \frac{d^m\theta}{(2\pi)^m}\, d^d X \;+\; O(\mathbf{g}^2)$$

Consider the Poisson brackets:

$$\{J'_{[\mathbf{q}]}, J'_{[\mathbf{p}]}\}|_{\mathcal{K}} \;=\; \left\{\int q_\nu(\mathbf{X})\, J'^{\nu}(\mathbf{X})\, d^dX \;,\; \int p_\mu(\mathbf{Y})\, J'^{\mu}(\mathbf{Y})\, d^dY\right\}\Big|_{\mathcal{K}} \;=$$

$$= \int q_\nu(\mathbf{X})\, \frac{\partial U'^{\nu}}{\partial U^{\lambda}}(\mathbf{X}) \left[\left\{J^{\lambda}(\mathbf{X}) \;,\; \int p_\mu(\mathbf{Y})\, J'^{\mu}(\mathbf{Y})\, d^dY\right\}\Big|_{\mathcal{K}} + O(\epsilon^2)\right] d^dX \;+$$

$$+ \int q_\nu(\mathbf{X}) \sum_q \Gamma^{\nu}_q(\mathbf{U}(\mathbf{X})) \left[\kappa^{(q)}_{i[\mathbf{U}(\mathbf{X})]}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}\right) + O(\epsilon)\right] \times$$

$$\times \left\{g^i(\boldsymbol{\theta}, \mathbf{X}) \,,\, J'^{\mu}(\mathbf{Y})\right\}\big|_{\mathcal{K}}\, p_\mu(\mathbf{Y})\, \frac{d^m\theta}{(2\pi)^m}\, d^dX\, d^dY$$

By Lemma 3.2$'$ we have the relation $\{g^i(\boldsymbol{\theta}, \mathbf{X}) \,,\, J'_{[\mathbf{p}]}\}|_{\mathcal{K}} = O(\epsilon)$ on the submanifold $\mathcal{K}$. In addition, completely analogous to relation (3.44) holds the relation

$$\int_0^{2\pi} \!\!\cdots \int_0^{2\pi} \kappa^{(q)}_{i[\mathbf{U}(\mathbf{X})]}\left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}\right) \left\{g^i(\boldsymbol{\theta}, \mathbf{X}) \,,\, J'_{[\mathbf{p}]}\right\}|_{\mathcal{K}[1]}\, \frac{d^m\theta}{(2\pi)^m} \;\equiv\; 0$$

by virtue of the original dependence of the constraints $g^i(\boldsymbol{\theta}, \mathbf{X})$. We thus obtain:

$$\iint q_\nu(\mathbf{X}) \;\; \{J'^{\nu}(\mathbf{X}) \,,\, J'^{\mu}(\mathbf{Y})\}|_{\mathcal{K}}\, p_\mu(\mathbf{Y})\, d^dX\, d^dY \;=$$

$$= \int q_\nu(\mathbf{X})\, \frac{\partial U'^{\nu}}{\partial U^{\lambda}}(\mathbf{X}) \left\{J^{\lambda}(\mathbf{X}) \,,\, \int p_\mu(\mathbf{Y})\, J'^{\mu}(\mathbf{Y})\, d^dY\right\}\Big|_{\mathcal{K}} d^dX \;+\; O(\epsilon^2)$$

Repeating the arguments for the functional $\int p_\mu(\mathbf{Y})\, J'^{\mu}(\mathbf{Y})\, d^dY$ we finally obtain

$$\iint q_\nu(\mathbf{X}) \;\; \{J'^{\nu}(\mathbf{X}) \,,\, J'^{\mu}(\mathbf{Y})\}|_{\mathcal{K}}\; p_\mu(\mathbf{Y})\; d^dX\, d^dY \;= \tag{3.84}$$

$$= \iint q_\nu(\mathbf{X}) \;\; \frac{\partial U'^{\nu}}{\partial U^{\lambda}}(\mathbf{X}) \;\; \{J^{\lambda}(\mathbf{X}) \,,\, J^{\sigma}(\mathbf{Y})\}\big|_{\mathcal{K}} \;\; \frac{\partial U'^{\mu}}{\partial U^{\sigma}}(\mathbf{Y})\; p_\mu(\mathbf{Y})\; d^dX\, d^dY \;+\; O(\epsilon^2)$$

Given that the principal (in ϵ) terms in the expressions $\{J^{\lambda}(\mathbf{X}) \,,\, J^{\sigma}(\mathbf{Y})\}|_{\mathcal{K}}$ and $\{J'^{\nu}(\mathbf{X}) \,,\, J'^{\mu}(\mathbf{Y})\}|_{\mathcal{K}}$ coincide with the Dubrovin - Novikov forms, obtained with the aid of the sets $(I^1, \dots, I^N)$ and $(I'^1, \dots, I'^N)$ respectively, we conclude from (3.84):

$$\{U'^{\nu}(\mathbf{X}) \,,\, U'^{\mu}(\mathbf{Y})\}'_{DN} \;=\; \frac{\partial U'^{\nu}}{\partial U^{\lambda}}(\mathbf{X}) \;\; \{U^{\lambda}(\mathbf{X}) \,,\, U^{\sigma}(\mathbf{Y})\}_{DN} \;\; \frac{\partial U'^{\mu}}{\partial U^{\sigma}}(\mathbf{Y})$$

which means the coinciding of the forms $\{\dots, \dots\}_{DN}$ and $\{\dots, \dots\}'_{DN}$.

As a result, we can claim that the relations

$$\left\{k^{\alpha}_p(\mathbf{U}(\mathbf{X})) \,,\, k^{\beta}_q(\mathbf{U}(\mathbf{Y}))\right\} \;=\; 0 \quad , \quad \left\{k^{\alpha}_p(\mathbf{U}(\mathbf{X})) \,,\, U^{\gamma}(\mathbf{Y})\right\} \;=\; \left[\omega^{\alpha\gamma}(\mathbf{U}(\mathbf{X}))\;\; \delta(\mathbf{X} - \mathbf{Y})\right]_{X^p}$$

$(\alpha, \beta = 1, \dots, m,\; \gamma = 1, \dots, m + s)$ and the expressions for

$$\{U^{\gamma}(\mathbf{X})\,,\,U^{\rho}(\mathbf{Y})\}_{DN} \quad , \quad \gamma,\rho\,=\,1,\dots,m+s$$

transform into the expressions

$$\left\{k_p^{\alpha}(\mathbf{U}'(\mathbf{X}))\,,\,k_q^{\beta}(\mathbf{U}'(\mathbf{Y}))\right\} \;=\; 0 \quad , \quad \left\{k_p^{\alpha}(\mathbf{U}'(\mathbf{X}))\,,\,U'^{\gamma}(\mathbf{Y})\right\} \;=\; \left[\omega'^{\alpha\gamma}(\mathbf{U}'(\mathbf{X}))\;\delta(\mathbf{X}-\mathbf{Y})\right]_{X^p}$$

and

$$\{U'^{\gamma}(\mathbf{X})\,,\,U'^{\rho}(\mathbf{Y})\}'_{DN}$$

after the change of coordinates

$$\left(U^1,\dots,U^N\right) \;\rightarrow\; \left(U'^1,\dots,U'^N\right)$$

on Λ. We can claim then that brackets (3.55) obtained with the aid of the sets $(I^1,\dots,I^N)$ and $(I'^1,\dots,I'^N)$ transform into each other after the change of coordinates

$$\left(S^1(\mathbf{X}),\dots,S^m(\mathbf{X}),\,U^1(\mathbf{X}),\dots,U^{m+s}(\mathbf{X})\right) \;\rightarrow\; \left(S^1(\mathbf{X}),\dots,S^m(\mathbf{X}),\,U'^1(\mathbf{X}),\dots,U'^{m+s}(\mathbf{X})\right)$$

where $U'^{\gamma}(\mathbf{X}) = U'^{\gamma}\left(\mathbf{S}_{\mathbf{X}},\,U^1(\mathbf{X}),\dots,U^{m+s}(\mathbf{X})\right)$.

Theorem 3.2 is proved.

Let us note that Theorem 3.2 means, in particular, that bracket (3.55) is also invariant with respect to the choice of the functionals $(I^1,\dots,I^{m+s})$ among the full set $\{I^{\nu},\ \nu=1,\dots,N\}$.

Finally, we prove the theorem about the Hamiltonian properties of the Whitham system (2.40) - (2.41) under the same conditions as before.

Theorem 3.3.
Let Λ be a regular Hamiltonian family of m-phase solutions of (2.5). Let $(I^1,\dots,I^N)$ be a complete Hamiltonian set of commuting first integrals of (2.5) having the form (2.8) and H be the Hamiltonian function for system (2.5) having the form (2.33). Then the Whitham system (2.40) - (2.41) is Hamiltonian with respect to the corresponding bracket (3.55) with the Hamiltonian function

$$H^{av} \;=\; \int \langle P_H\rangle \left(\mathbf{S}_{\mathbf{X}},\,U^1(\mathbf{X}),\dots,U^{m+s}(\mathbf{X})\right)\; d^dX$$

Proof.
By Theorem 3.2, without loss of generality we can assume that the Hamiltonian functional H belongs to the set $(I^1,\dots,I^{m+s})$, $H=I^{\mu_0}$. It is easy to verify then that the corresponding Hamiltonian H^{av} generates in this case the system

$$S_T^{\alpha} \;=\; \omega^{\alpha\mu_0}\left(\mathbf{S}_{\mathbf{X}},\,U^1,\dots,U^{m+s}\right) \quad , \quad \alpha=1,\dots,m$$

$$U_T^{\gamma} \;=\; \langle Q^{\gamma\mu_0 1}\rangle_{X^1} \;+\; \dots \;+\; \langle Q^{\gamma\mu_0 d}\rangle_{X^d} \quad , \quad \gamma=1,\dots,m+s$$

i.e. exactly system (2.40) - (2.41).

Theorem 3.3 is proved.

4 On the canonical forms of the averaged brackets.

Let us consider now "canonical forms" acquired by the averaged brackets in some special coordinates. We first prove here the following lemma:

Lemma 4.1.
Consider the averaged bracket (3.55) in the coordinates $(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}),\, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}))$. *There exists locally an invertible change of coordinates*

$$\left(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}),\, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})\right) \;\rightarrow$$
$$\rightarrow\; \left(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}),\, Q_1(\mathbf{X}), \dots, Q_m(\mathbf{X}),\, N^1(\mathbf{X}), \dots, N^l(\mathbf{X})\right)$$

where

$$Q_\alpha \;=\; Q_\alpha\left(\mathbf{S_X},\, U^1, \dots, U^{m+s}\right) \quad , \quad N^l \;=\; N^l\left(\mathbf{S_X},\, U^1, \dots, U^{m+s}\right)$$

such that the bracket (3.55) has in the new coordinates the form:

$$\left\{S^\alpha(\mathbf{X})\,,\,S^\beta(\mathbf{Y})\right\} \;=\; 0 \;\;,\;\; \left\{S^\alpha(\mathbf{X})\,,\,Q_\beta(\mathbf{Y})\right\} \;=\; \delta^\alpha_\beta\,\delta(\mathbf{X}-\mathbf{Y}) \;\;,\;\; \left\{S^\alpha(\mathbf{X})\,,\,N^l(\mathbf{Y})\right\} \;=\; 0 \tag{4.1}$$

$$\left\{Q_\alpha(\mathbf{X})\,,\,Q_\beta(\mathbf{Y})\right\} \;=\; J_{\alpha\beta}\,[\mathbf{S},\mathbf{N}]\,(\mathbf{X}-\mathbf{Y}) \quad , \quad \left\{Q_\alpha(\mathbf{X})\,,\,N^l(\mathbf{Y})\right\} \;=\; J^l_\alpha\,[\mathbf{S},\mathbf{N}]\,(\mathbf{X}-\mathbf{Y})$$
$$\left\{N^l(\mathbf{X})\,,\,N^q(\mathbf{Y})\right\} \;=\; J^{lq}\,[\mathbf{S},\mathbf{N}]\,(\mathbf{X}-\mathbf{Y}) \tag{4.2}$$

($\alpha,\beta = 1,\dots,m\,,\; l,q = 1,\dots,s$), where $J_{\alpha\beta}$, J^l_α, J^{lq} are local distributions of the gradation degree 1 given by linear combinations of the function $\delta(\mathbf{X}-\mathbf{Y})$ and its first derivatives with local coefficients, depending on the values $(\mathbf{S_X},\, \mathbf{N}(\mathbf{X}),\, \mathbf{S_{XX}},\, \mathbf{N_X}(\mathbf{X}))$.

Proof.

Let us fix the parameters $k^\alpha_p = S^\alpha_{X^p}$ on the family Λ and consider the coordinates $\mathbf{U} = (U^1,\, \dots,\, U^{m+s})$ on the $(m+s)$-dimensional submanifolds $k^\alpha_p = \text{const}$ (for all α, p). Let us consider the vector fields

$$\vec{\xi}_{(\alpha)} \;=\; \left(\omega^{\alpha\,1}(\mathbf{S_X},\,\mathbf{U}), \dots,\, \omega^{\alpha\,m+s}(\mathbf{S_X},\,\mathbf{U})\,\right)^t$$

on the submanifolds $k^\alpha_p = \text{const}$.

From the Jacobi identities

$$\left\{\left\{U^\nu(\mathbf{X})\,,\,S^\alpha(\mathbf{Y})\right\}\,,\,S^\beta(\mathbf{Z})\right\} \;-\; \left\{\left\{U^\nu(\mathbf{X})\,,\,S^\beta(\mathbf{Z})\right\}\,,\,S^\alpha(\mathbf{Y})\right\} \;\equiv\; 0$$

for the bracket (3.55) it is easy to get the relations

$$\left[\,\vec{\xi}_{(\alpha)}\,,\,\vec{\xi}_{(\beta)}\,\right] \;\equiv\; 0 \quad , \quad \alpha,\beta = 1,\dots,m$$

for the commutators of the vectors fields $\vec{\xi}_{(\alpha)}$ on the submanifolds $k^\alpha_p = \text{const}$.

According to relations (3.2) for the frequencies generated by the functionals $(I^1, \dots, I^{m+s})$ we can state also that the set $\{\vec{\xi}_{(\alpha)}\}$ is linearly independent at every point.

We can claim then that on every submanifold

$$\{k^{\alpha}_{p} = \mathrm{const} \ \ , \ \ \alpha = 1, \dots, m \ , \ p = 1, \dots, d\}$$

there exists locally an invertible change of coordinates

$$\left(U^1, \dots, U^{m+s}\right) \ \rightarrow \ \left(Q_1(\mathbf{S_X}, \mathbf{U}), \dots, Q_m(\mathbf{S_X}, \mathbf{U}), \ N^1(\mathbf{S_X}, \mathbf{U}), \dots, N^s(\mathbf{S_X}, \mathbf{U}) \right)$$

depending smoothly on the parameters $k^{\alpha}_{p} = S^{\alpha}_{X^p}$, such that the coordinate representation of the vector fields $\vec{\xi}_{(\alpha)}$ on these submanifolds has the form

$$\vec{\xi}_{(1)} \ = \ (1, 0, \dots, 0)^t \ , \quad \dots \quad , \ \vec{\xi}_{(m)} \ = \ (0, \dots, 0, 1, 0, \dots, 0)^t$$

It is not difficult to see then that the corresponding change of coordinates

$$(\, \mathbf{S}(\mathbf{X})\, ,\, \mathbf{U}(\mathbf{X})\,) \ \ \rightarrow \ \ (\, \mathbf{S}(\mathbf{X})\, ,\, \mathbf{Q}(\mathbf{X})\, ,\, \mathbf{N}(\mathbf{X})\,)$$

provides relations (4.1) for bracket (3.55).

The general form of the distributions $J_{\alpha\beta}$, J^{l}_{α}, J^{lq} follows then from the form of bracket (3.55) and for the proof of the Lemma we have just to prove the relations

$$\delta J_{\alpha\beta} \, / \, \delta Q_{\gamma}(\mathbf{Z}) \ \equiv \ 0 \quad , \quad \delta J^{l}_{\alpha} \, / \, \delta Q_{\gamma}(\mathbf{Z}) \ \equiv \ 0 \quad , \quad \delta J^{lq} \, / \, \delta Q_{\gamma}(\mathbf{Z}) \ \equiv \ 0 \quad , \quad \gamma = 1, \dots, m \qquad (4.3)$$

for the functionals $J_{\alpha\beta}$, J^{l}_{α}, J^{lq}.

Using the Jacobi identities

$$\{\{Q_{\alpha}(\mathbf{X})\, ,\, Q_{\beta}(\mathbf{Y})\} \, , \, S^{\gamma}(\mathbf{Z})\} \ + \ c.p. \ \equiv \ 0 \quad , \quad \left\{\left\{Q_{\alpha}(\mathbf{X})\, ,\, N^{l}(\mathbf{Y})\right\} \, , \, S^{\gamma}(\mathbf{Z})\right\} \ + \ c.p. \ \equiv \ 0$$

$$\left\{\left\{N^{l}(\mathbf{X})\, ,\, N^{q}(\mathbf{Y})\right\} \, , \, S^{\gamma}(\mathbf{Z})\right\} \ + \ c.p. \ \equiv \ 0$$

we then easily get relations (4.3).

Lemma 4.1 is proved.

Let us note here that according to their definition the coordinates $(Q_{\alpha}\, ,\, N^{l})$ are defined modulo the transformations

$$Q_{\alpha} \ \rightarrow \ Q_{\alpha} \, + \, q_{\alpha}(\mathbf{S_X},\, \mathbf{N}) \quad , \quad N^{l} \ \rightarrow \ \tilde{N}^{l}(\mathbf{S_X},\, \mathbf{N})$$

where the transformation $N^{l} \ \rightarrow \ \tilde{N}^{l}(\mathbf{S_X},\, \mathbf{N})$ is invertible for every fixed $\mathbf{S_X}$.

Let us consider now a special case, when the additional parameters $(n^1, \dots, n^s)$ are absent on the family Λ and the full regular family of m-phase solutions of system (2.5) can be parametrized by the $m(d+1)$ parameters $(k^{\alpha}_{p}, \omega^{\alpha})$, $\alpha = 1, \dots, m$, $p = 1, \dots, d$.

In this situation we can prove the following theorem about the canonical form of the averaged bracket (3.55).

Theorem 4.1.

Let system (2.5) be a local Hamiltonian system generated by the functional (2.33) in the local field-theoretic Hamiltonian structure (2.32). Let Λ be a regular Hamiltonian family of m-phase solutions of (2.5) and $(I^1, \dots, I^{m(d+1)})$ be a complete Hamiltonian set of commuting integrals (2.8) for this family.

Let relations (3.55) represent a Poisson bracket on the space of fields

$$(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}),\, U^1(\mathbf{X}), \dots, U^m(\mathbf{X}))$$

Then locally there exists an invertible change of coordinates

$$\big(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}),\, U^1(\mathbf{X}), \dots, U^m(\mathbf{X})\big) \;\rightarrow\; \big(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}),\, Q_1(\mathbf{X}), \dots, Q_m(\mathbf{X})\big)$$

where $Q_\alpha \;=\; Q_\alpha(\mathbf{S_X},\, \mathbf{U})$ such that the bracket (3.55) acquires in the coordinates $(\mathbf{S}(\mathbf{X})\,,\, \mathbf{Q}(\mathbf{X}))$ the following non-degenerate canonical form:

$$\big\{S^\alpha(\mathbf{X})\,,\, S^\beta(\mathbf{Y})\big\} \;=\; 0 \quad , \quad \{S^\alpha(\mathbf{X})\,,\, Q_\beta(\mathbf{Y})\} \;=\; \delta^\alpha_\beta\; \delta(\mathbf{X}-\mathbf{Y}) \quad , \quad \{Q_\alpha(\mathbf{X})\,,\, Q_\beta(\mathbf{Y})\} \;=\; 0$$

Proof.

Let us give here just a proof in abstract field-theoretical form. The same considerations can be made also in the differential-geometric form after concrete substitutions for the coordinate transformations. However, we would like to skip here the corresponding calculations to avoid rather bulky expressions in the text.

In accordance with Lemma 4.1 let us write down the averaged bracket (3.55) in coordinates $(S^\alpha(\mathbf{X})\,,\, \tilde{Q}_\alpha(\mathbf{X}))$ such that

$$\big\{S^\alpha(\mathbf{X})\,,\, S^\beta(\mathbf{Y})\big\} \;=\; 0 \quad , \quad \Big\{S^\alpha(\mathbf{X})\,,\, \tilde{Q}_\beta(\mathbf{Y})\Big\} \;=\; \delta^\alpha_\beta\; \delta(\mathbf{X}-\mathbf{Y})$$

$$\Big\{\tilde{Q}_\alpha(\mathbf{X})\,,\, \tilde{Q}_\beta(\mathbf{Y})\Big\} \;=\; \tilde{J}_{\alpha\beta}\,[\mathbf{S}]\,(\mathbf{X}-\mathbf{Y})$$

From the Jacobi identities

$$\Big\{\Big\{\tilde{Q}_\alpha(\mathbf{X})\,,\, \tilde{Q}_\beta(\mathbf{Y})\Big\}\,,\, \tilde{Q}_\gamma(\mathbf{Z})\Big\} \;+\; c.p. \;\equiv\; 0$$

we then easily get the relations

$$\frac{\delta \tilde{J}_{\alpha\beta}[\mathbf{S}](\mathbf{X},\mathbf{Y})}{\delta S^\gamma(\mathbf{Z})} \;+\; \frac{\delta \tilde{J}_{\beta\gamma}[\mathbf{S}](\mathbf{Y},\mathbf{Z})}{\delta S^\alpha(\mathbf{X})} \;+\; \frac{\delta \tilde{J}_{\gamma\alpha}[\mathbf{S}](\mathbf{Z},\mathbf{X})}{\delta S^\beta(\mathbf{Y})} \;\equiv\; 0 \tag{4.4}$$

for the form $\tilde{J}_{\alpha\beta}[\mathbf{S}](\mathbf{X},\mathbf{Y}) \;\equiv\; \tilde{J}_{\alpha\beta}[\mathbf{S}](\mathbf{X}-\mathbf{Y})$.

Relations (4.4) mean that the 2-form $\tilde{J}_{\alpha\beta}[\mathbf{S}](\mathbf{X},\mathbf{Y})$ on the space of fields $(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}))$ is closed and can be locally represented as

$$\tilde{J}_{\alpha\beta}[\mathbf{S}](\mathbf{X},\mathbf{Y}) \;=\; \frac{\delta q_\alpha[\mathbf{S}](\mathbf{X})}{\delta S^\beta(\mathbf{Y})} \;-\; \frac{\delta q_\beta[\mathbf{S}](\mathbf{Y})}{\delta S^\alpha(\mathbf{X})}$$

for some 1-form $q_\alpha[\mathbf{S}](\mathbf{X})$ on the same phase space. Easy to see now that the change

$$Q_\alpha(\mathbf{X}) \;\; = \;\; \tilde{Q}_\alpha(\mathbf{X}) \; - \; q_\alpha[\mathbf{S}](\mathbf{X})$$

gives the necessary coordinates $Q_\alpha(\mathbf{X})$.

From the form of the functionals $\tilde{J}_{\alpha\beta}[\mathbf{S}](\mathbf{X},\mathbf{Y})$ it's not difficult to show also that the 1-form $q_\alpha[\mathbf{S}](\mathbf{X})$ can be chosen in the form $q_\alpha[\mathbf{S}](\mathbf{X}) \equiv q_\alpha(\mathbf{S}_\mathbf{X})$ with some smooth functions $q_\alpha(\mathbf{S}_\mathbf{X})$, which gives the necessary form of the corresponding coordinate transformation.

Theorem 4.1 is proved.

Let us note that in more general case of the presence of additional parameters $(n^1,\dots,n^s)$ we can expect in fact, that the bracket (4.1) - (4.2) can have more complicated, maybe even degenerate, form.

We can note also that the bracket $\{N^l(\mathbf{X})\,,\,N^q(\mathbf{Y})\}$ given by (4.2) represents a Poisson bracket on the space of fields $\mathbf{N}(\mathbf{X})$ for any fixed values of $\mathbf{S}(\mathbf{X})$ as follows from the form of the bracket (4.1) - (4.2). This actually gives rather strong restrictions on the form of the bracket (4.2).

At last we consider just a very simple example of the averaging of a non-degenerate bracket in the single-phase case. Let us consider the nonlinear wave equation in d spatial dimensions having the form

$$\varphi_{tt} \; - \; \Delta\,\varphi \;\; = \;\; V^\prime(\varphi) \tag{4.5}$$

with some potential function $V(\varphi)$.

The periodic one-phase solutions of (4.5) are defined by the equation

$$\left(\omega^2 - k_1^2 - \cdots - k_d^2\right)\,\varphi_{\theta\theta} \;\; = \;\; V^\prime(\varphi) \tag{4.6}$$

which implies

$$\frac{\omega^2 - k_1^2 - \cdots - k_d^2}{2}\;\varphi_\theta^2 \;\; = \;\; V(\varphi) \; + \; C$$

with some integration constant C. We can assume for example that we have here $\omega^2 \; > \; \mathbf{k}^2$ though it does not affect the final conclusions.

The solutions $\varphi(\theta,\,\omega,k_1,\dots,k_d)$ are then given by the quadrature

$$\pm\sqrt{\frac{\omega^2 - k_1^2 - \cdots - k_d^2}{2}}\;\int\frac{d\,\varphi}{\sqrt{V(\varphi)+C}} \;\; = \;\; \theta \; + \; \theta_0$$

where the value of φ oscillates between two subsequent zeros of the expression $V(\varphi)+C$, while the condition

$$\sqrt{\frac{\omega^2 - k_1^2 - \cdots - k_d^2}{2}}\;\oint\frac{d\,\varphi}{\sqrt{V(\varphi)+C}} \;\; = \;\; 2\pi$$

fixes the value of the integration constant C.

Let us denote the corresponding function $\varphi(\theta,\,\omega,k_1,\dots,k_d)$ as $\Phi(\theta+\theta_0,\,\omega,k_1,\dots,k_d)$.

Equation (4.5) can be written in the Hamiltonian form

$$\varphi_t = \psi \quad , \quad \psi_t = \Delta\varphi + V'(\varphi) \tag{4.7}$$

with the Hamiltonian function

$$H \equiv \int P_H(\varphi, \psi, \dots)\, d^d x \equiv \int \left(\frac{\psi^2}{2} + \frac{(\nabla\varphi)^2}{2} - V(\varphi) \right) d^d x \tag{4.8}$$

and the Poisson bracket

$$\{\varphi(\mathbf{x}), \varphi(\mathbf{y})\} = 0 \quad , \quad \{\varphi(\mathbf{x}), \psi(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}) \quad , \quad \{\psi(\mathbf{x}), \psi(\mathbf{y})\} = 0$$

The family Λ is represented then by the family of the functions

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \Phi(\theta + \theta_0, \omega, k_1, \dots, k_d) \\ \omega\, \Phi_\theta(\theta + \theta_0, \omega, k_1, \dots, k_d) \end{pmatrix}$$

parametrized by the values $(\omega, k_1, \dots, k_d)$ and the initial phase θ_0.

Using equation (4.6) it is not difficult to show that the family Λ is a complete regular family of single-phase solutions of (4.7). The set of the commuting functionals $(I^1, \dots, I^{d+1})$ is given here by the momentum functionals

$$I_q \equiv \int P_q(\varphi, \psi, \dots)\, d^d x \equiv \int \varphi_{x^q}\, \psi\, d^d x \quad , \quad q = 1, \dots, d$$

and the Hamiltonian functional (4.8), which give a complete Hamiltonian set of the first integrals for the family Λ. The values of the averaged densities $\langle P_q \rangle$ on Λ are equal to

$$\langle P_q \rangle = k_q\, \omega \int_0^{2\pi} \Phi_\theta^2 \frac{d\theta}{2\pi} = k_q\, Q$$

where

$$Q \equiv \omega \int_0^{2\pi} \Phi_\theta^2 \frac{d\theta}{2\pi}$$

while the functionals I_q generate on Λ linear shifts of θ_0 with the frequencies $\omega_q = k_q$.

The pairwise Poisson brackets of the densities $P_q(\mathbf{x})$, $P_H(\mathbf{x})$ can be written in the form

$$\{P_q(\mathbf{x}), P_l(\mathbf{y})\} = P_q(\mathbf{x})\, \delta_{x^l}(\mathbf{x}-\mathbf{y}) + P_l(\mathbf{x})\, \delta_{x^q}(\mathbf{x}-\mathbf{y}) + [P_q(\mathbf{x})]_{x^l}\, \delta(\mathbf{x}-\mathbf{y}) \tag{4.9}$$

$$\{P_q(\mathbf{x}), P_H(\mathbf{y})\} = \psi^2(\mathbf{x})\, \delta_{x^q}(\mathbf{x}-\mathbf{y}) + \varphi_{x^q} \sum_l \varphi_{x^l}\, \delta_{x^l}(\mathbf{x}-\mathbf{y}) +$$
$$+ \left[\frac{1}{2}\psi^2 - \frac{1}{2}(\nabla\varphi)^2 + V(\varphi) \right]_{x^q} \delta(\mathbf{x}-\mathbf{y}) + \sum_l (\varphi_{x^q}\varphi_{x^l})_{x^l}\, \delta(\mathbf{x}-\mathbf{y}) \tag{4.10}$$

$$\{P_H(\mathbf{x}), P_H(\mathbf{y})\} = 2 \sum_l P_l(\mathbf{x})\, \delta_{x^l}(\mathbf{x}-\mathbf{y}) + \sum_l [P_l(\mathbf{x})]_{x^l}\, \delta(\mathbf{x}-\mathbf{y}) \tag{4.11}$$

Thus, if we choose the coordinates $S(\mathbf{X})$, $U_{(q)}(\mathbf{X}) = \langle P_q \rangle(\mathbf{X})$ for some $q = 1, \dots, d$, the corresponding bracket (3.82) will be written in the form:

$$\{S(\mathbf{X})\,,\,S(\mathbf{Y})\} \;=\; 0 \quad , \quad \left\{S(\mathbf{X})\,,\,U_{(q)}(\mathbf{Y})\right\} \;=\; S_{X^q}\;\delta(\mathbf{X}-\mathbf{Y})$$

$$\left\{U_{(q)}(\mathbf{X})\,,\,U_{(q)}(\mathbf{Y})\right\} \;=\; 2\,U_{(q)}(\mathbf{X})\;\delta_{X^q}(\mathbf{X}-\mathbf{Y}) \;+\; U_{(q)\,X^q}\;\delta(\mathbf{X}-\mathbf{Y}) \tag{4.12}$$

It's not difficult to check that in the coordinates

$$S(\mathbf{X}) \quad , \quad Q(\mathbf{X}) \;=\; \langle P_1\rangle/S_{X^1} \;=\; \dots \;=\; \langle P_d\rangle/S_{X^d}$$

any of the brackets (4.12) takes the form

$$\{S(\mathbf{X})\,,\,S(\mathbf{Y})\} \;=\; 0 \quad , \quad \{S(\mathbf{X})\,,\,Q(\mathbf{Y})\} \;=\; \delta(\mathbf{X}-\mathbf{Y}) \quad , \quad \{Q(\mathbf{X})\,,\,Q(\mathbf{Y})\} \;=\; 0 \tag{4.13}$$

We can see then that all the brackets (4.12) represent in fact the same bracket (4.13) in different coordinates. Thus, we can claim that all the brackets (4.12) transform into each other after the corresponding change of coordinates $U_{(q)}(\mathbf{X}) \;=\; S_{X^q}\,U_{(l)}(\mathbf{X})/S_{X^l}$.

The averaged density $\langle P_H\rangle$ on Λ can be represented in the form

$$\langle P_H\rangle \;=\; \int_0^{2\pi} \left[\frac{\omega^2 + k_1^2 + \cdots + k_d^2}{2}\;\Phi_\theta^2 \;-\; V\left(\Phi\right)\right]\,\frac{d\theta}{2\pi} \;=$$

$$=\; \omega^2 \int_0^{2\pi} \Phi_\theta^2\,\frac{d\theta}{2\pi} \;-\; \int_0^{2\pi} \left[\frac{\omega^2 - k_1^2 - \cdots - k_d^2}{2}\;\Phi_\theta^2 \;+\; V\left(\Phi\right)\right]\,\frac{d\theta}{2\pi}$$

Using the fact that the variation derivative of the second integral is equal to zero on Λ according to (4.6) it is not difficult to obtain the relation

$$d\,\langle P_H\rangle \;=\; \omega\,dQ \;+\; Q\sum_l \frac{k_l}{\omega}\;dk_l \tag{4.14}$$

From (4.14) it is easy to get the following relations in the bracket (4.13):

$$\{S(\mathbf{X})\,,\,\langle P_H\rangle(\mathbf{Y})\} \;=\; \omega\,(S_{\mathbf{X}},\,Q)\;\delta(\mathbf{X}-\mathbf{Y})$$

$$\{\langle P_H\rangle(\mathbf{X})\,,\,\langle P_H\rangle(\mathbf{Y})\} \;=\; 2\sum_l \langle P_l\rangle(\mathbf{X})\;\delta_{X^l}(\mathbf{X}-\mathbf{Y}) \;+\; \sum_l \langle P_l\rangle_{X^l}\;\delta(\mathbf{X}-\mathbf{Y})$$

From expression (4.11) for the pairwise Poisson brackets of the functionals $P_H(\mathbf{x})$, $P_H(\mathbf{y})$ we can see then that the bracket (3.82) obtained with the aid of the functional H coincides with the bracket (4.13) (and all (4.12)) after the corresponding change of coordinates.

The Whitham equations can be easily written using the averaged Poisson bracket and the functional $H^{av} = \int \langle P_H\rangle\,d^dX$. Thus, in the coordinates $(S(\mathbf{X}),\,Q(\mathbf{X}))$ the Whitham system takes the form

$$S_T \;=\; \omega\,(S_{\mathbf{X}},\,Q) \quad , \quad Q_T \;=\; \sum_l \left(\frac{S_{X^l}}{\omega}\;Q\right)_{X^l}$$

In conclusion the author expresses his gratitude to Prof. M.V. Pavlov for the interest to this work and fruitful discussions.

This work was financially supported by the Russian Federation Government Grant No. 2010-220-01-077, Grant of the President of Russian Federation NSh-4995.2012.1, and Grant RFBR No. 11-01-12067-ofi-m-2011.

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